Markovian queueing networks (cont')

COMPSCI 655
Lecture 12
Seen lecture 11: Open Jackson networks

\[ X_i(t) = \text{nb. customers time } t \text{ in station } i=1,\ldots,N \]

If eqns
\[ \lambda_i = \lambda_i^0 + \sum_{j=1}^{K} \lambda_j p(j,i) \quad \text{for } i=1,\ldots,K \]

have unique sol. such that \( 0 < \lambda_i < \mu_i \) for all \( i=1,\ldots,K \)

then, for all \( n_1 \geq 0, \ldots, n_K \geq 0 \), product-form result

\[
\lim_{n \to \infty} P(X_1(t)=n_1, \ldots, X_K(t)=n_K) = \lim_{n \to \infty} \prod_{1 \leq i \leq K} P(X_i(t)=n_i) = \prod_{i=1}^{K} (1 - \rho_i) \rho_i^{n_i}
\]

with \( \rho_i := \lambda_i / \mu_i < 1 \) for all \( i=1,\ldots,k \)
Stationary analysis of closed Jackson networks

\( X_i(t) = 0,1, \ldots \) state of node \( i = 1, \ldots, K \) at time \( t \)

\( X(t) = (X_1(t), \ldots, X_K(t)) \) state of network at time \( t \)

with \( X_1(t) + \ldots + X_K(t) = N \)

Claim: \( \{X(t), t \geq 0\} \) is a CTMC with state-space \( S(N,K) := \{(n_1, \ldots, n_K) \in \{0,1,\ldots,N\}^K, n_1 + \ldots + n_K = N\} \)

Proof. Use rule of construction for CTMC
Stationary analysis of closed Jackson networks (con’t)

Non-zero transitions from state \( n = (n_1, \ldots, n_K) \):

- \( n \rightarrow n - e_i + e_j \) customer leaving station \( i \) enters station \( j \neq i \) (provided \( n_i > 0 \) or equivalently \( n_j < N \))

\( K(K-1) \) exponential independent rvs with rates

\( \mu_i p(i,j) 1(n_i > 0) \) for \( i=1,\ldots,K, j=1,\ldots,K, i \neq j \)

- Rule of construction applies and yields

\[
q(n, n-e_i + e_j) = \mu_i p(i,j) 1(n_i > 0) \text{ for } i \neq j
\]
Stationary analysis of closed Jackson networks (con't)

\[ q(n, n-e_i+e_j) = \mu_i p(i, j) 1(n_i > 0), \ i \neq j, \text{ non-zero transitions} \]

Balance eqns:

\[ \pi(n) \sum_{i=1}^{K} \mu_i 1(n_i > 0)(1 - p(i,i)) = \sum_{i=1}^{K} \sum_{j=1 \atop j \neq i}^{K} \pi(n+e_i-e_j) \mu_i 1(n_i > 0)p(i,j) \]

Easy to see that \[ \pi(n) = \prod_{i=1}^{K} \left( \frac{\lambda_i}{\mu_i} \right)^{n_i} \] with \( \lambda_1, ..., \lambda_K \) any non-zero solution (irreducibility cond.) of eqns

\[ \lambda_i = \sum_{1 \leq j \leq K} p(j, i) \lambda_j \quad i=1, ..., K \]
solves the balance eqns
Result (Limiting distribution closed Jackson network)

If $\lambda_1, \ldots, \lambda_K$ is any non-zero solution (irreducibility) of eqns $\lambda_i = \sum_{1 \leq j \leq K} p(j,i) \lambda_j$, $i=1,\ldots,K$, then

$$
\pi(n) = \frac{1}{G(N,K)} \prod_{i=1}^{K} \rho_i^{n_i} \quad \text{for all } n \in S(N,K)
$$

is the limiting distribution of joint number of customers network (i.e. $\lim_{t \to \infty} P(X(t)=n) = \pi(n)$ for all $n$) regardless of initial state, where

$$
G(N,K) = \sum_{n \in S(N,K)} \prod_{i=1}^{K} \rho_i^{n_i}
$$
Comments

1) Not a product-form result. Not surprising as rvs $X_1, \ldots, X_K$ cannot be independent as they sum up to N.

2) Can be an issue to calculate normalizing constant

$$G(N,K) = \sum_{n \in S(N,K)} \prod_{i=1}^{K} \rho_i^{n_i} \quad \text{as} \quad |S(N,K)| = (N+K-1)!/(N!(K-1)!)$$

Ex.: K=10 (10 queues) and N=35 (35 customers)
Then, S(N,K) has 52,451,256 terms ...
Performance measures

\[ P(X_i \geq k) = \frac{1}{G(N, K)} \sum_{n \in S(N, K)} \prod_{i=1}^{K} \rho_i^{n_i} = \rho_i^k \frac{1}{G(N, K)} \sum_{n \in S(N-k, K)} \prod_{i=1}^{K} \rho_i^{n_i} \]

\[ = \rho_i^k \frac{G(N-k, K)}{G(N, K)} \]

\[ E[X_i] = \sum_{1 \leq k \leq K} k P(X_i = k) = \sum_{1 \leq k \leq K} P(X_i \geq k) \]

\[ = \frac{1}{G(N, K)} \sum_{k=1}^{N} \rho_i^k G(N-k, K) \]
Performance measures (con’t)

- $P(X_i = k) = P(X_i \geq k) - P(X_i \geq k+1)$
  \[= \rho_i^k \frac{1}{G(N, K)} (G(N - k, K) - \rho_i G(N - k - 1, K))\]

- Utilization node $i$
  \[U_i = P(X_i > 0) = 1 - \frac{1}{G(N, K)} (G(N - 1, K) - \rho_i G(N - 1, K)) = \rho_i \frac{G(N - 1, K)}{G(N, K)}\]

- Throughput node $i$
  \[T_i = \mu_i P(X_i > 0) = \lambda_i \frac{G(N - 1, K)}{G(N, K)}\]
Need to devise an efficient algorithm to compute normalizing constant $G(N,K)$ as all performance metrics express in terms of $G(1,K)$, $G(2,K)$, ..., $G(N,K)$

One answer: Buzen’s convolution algorithm
Buzen’s convolution algorithm

\[ G(n,m) = \sum_{n \in S(n,m)} \prod_{i=1}^{m} \rho_i^{n_i} = \sum_{k=0}^{n} \sum_{n \in S(n,m)} \prod_{i=1}^{m} \rho_i^{n_i} = \sum_{k=0}^{n} \rho_m^k \sum_{n \in S(n-k,m-1)} \prod_{i=1}^{m-1} \rho_i^{n_i} \]

= \sum_{k=0}^{n} \rho_m^k G(n-k,m-1)

with initial conditions \( G(n,1) = \rho_1^n \) for \( n=0,1,\ldots,N \)
\( G(0,m) = 1 \) for \( m=1,\ldots,N \)

Complexity of \( NK \)
Ex. 4: Closed central server

N jobs
K=D+T+1 nodes
(1 server,
D secondary devices,
T users)

\( \lambda_0 = \lambda_1 + \ldots + \lambda_{D+T} \)
\( \lambda_i = \lambda_0 p r_i, \ i=1,\ldots,D \)
\( \lambda_i = \lambda_0 (1-p) q_{i-D}, i=D+1,\ldots,D+T \)

\( 0<p<1, \ q_i > 0, \ r_j > 0, \ i=1,\ldots,T, \ j=1,\ldots,D \)
Ex. 4: Closed central server (cont’)

\[
\pi(n) = \frac{1}{G(N, D+T+1)} \prod_{i=0}^{D+T} \rho_i^n \quad \text{for all } n \in S(M, D+T+1)
\]  

(1)

with \( \rho_0 = \frac{1}{\mu_0} \)

\[
\rho_i = \frac{pr_i}{\mu_i} \quad \text{for } i=1,\ldots,D
\]

\[
\rho_i = \frac{(1-p)q_{i-D}}{\mu_i} \quad \text{for } i=D+1,\ldots,D+T
\]

Throughput server : \( \frac{G(M-1,D+T+1)}{G(M,D+T+1)} \)

Note: Result correct if for some \( i=1,\ldots,D \) \( r_i = 0 \) and/or for some \( j=1,\ldots,T \) \( q_j = 0 \) as (1) gives \( P(X_i = 0) = 1 \) and \( P(X_j = 0) = 1 \) (in either case the CTMC is not irreducible). Same with \( p=0 \) or \( p=1 \).
Extensions of Jackson networks

- **Mixed networks** (notion of subchains)
- **M/M/c nodes** (instead of M/M/1) for both open, closed and mixed networks

More generally, exponential service times with rate depending on node occupation: $\mu_i(n)$ when $X_i(t)=n$

- **Classes** (while travelling through the network customers may change class): customer of class $r$ leaving station $i$ joins station $j$ as customer of class $s$ with probability $p(i,r;j,s)$

Ex.: customers of same class follow same path

See Proposition 22 in Lecture Notes (pp. 43-44) and many more extensions are possible ....
Extensions of Jackson networks (cont’)

- **PS** (Processor Sharing), **LIFO** (Last-in-First-out), **IS** (Infinitely many Servers) nodes

Service time distribution at these nodes may be arbitrary and may depend on the customer’s class

→ **BCMP** (Basket, Chandy, Muntz, Palacios) networks (1975)

See Theorem 1 in Lecture Notes (pp. 49-50)

- State-dependent routing
- State-dependent exogenous arrival rates
- Etc.
Toward BCMP: Multiclass PS node

- Single server, infinite waiting room
- Each customer receives same fraction of server’s power: if n customers, each gets 1/n
- R classes of customers: Assume for time being that service times of class-r customers is exponentially distributed with rate $\mu_r$
- Poisson arrival with rate $\lambda_r$ for class-r
- All service times and arrival processes independent

Set $\rho_r = \lambda_r / \mu_r$ and $\rho = \sum_{1 \leq r \leq R} \rho_r$
\{X(t)>>=(X_1(t), \ldots, X_R(t))>>>, t \geq 0\}, with \(X_r(t) = \# \text{ class}-r \text{ in node at time } t\), is a CTMC

Transitions from state \(n=(n_1,\ldots,n_R)\):
- \(n \rightarrow n + e_r : \lambda_r \)
- \(n \rightarrow n - e_r : \mu_r n_r 1(n_r>0)/|n|\) with \(|n|:=n_1 + \ldots + n_R\)

\[
\pi(n) \sum_{r=1}^{R} \left( \lambda_r + \mu_r \frac{n_r}{|n|} \right) = \sum_{r=1}^{R} \pi(n - e_r) \lambda_r 1(n_r > 0) + \sum_{r=1}^{R} \pi(n + e_r) \mu_r \frac{n_r + 1}{|n + e_r|}
\]

Easy to check \(|n|! \prod_{r=1}^{R} \left( \frac{\lambda_r}{\mu_r} \right)^{n_r} \frac{n_r!}{n_r!} \) solves balance eqns
Normalizing constant

\[ \pi(n) = \frac{|n|!}{G} \prod_{r=1}^{R} \rho_r^{n_r} \] with \( G := \sum_{n_1 \geq 0, \ldots, n_R \geq 0} |n|! \prod_{r=1}^{R} \rho_r^{n_r} / n_r! \)

Multinomial distr.

\[ G = \sum_{n_1 \geq 0, \ldots, n_R \geq 0} |n|! \prod_{r=1}^{R} \rho_r^{n_r} / n_r! = \sum_{n \geq 0} \sum_{n_1 + \ldots + n_R = n} |n|! \prod_{r=1}^{R} \rho_r^{n_r} / n_r! \]

\[ = \sum_{n \geq 0} (\rho_1 + \ldots + \rho_R)^n = (1 - \rho)^{-1} \]

so that \( \pi(n) = (1 - \rho)|n|! \prod_{r=1}^{R} \rho_r^{n_r} / n_r! \)
Toward BCMP: Multiclass PS node (con’t)

Result (Limiting distribution multiclass PS node)

If \( \rho < 1 \) (stability)

then \( \pi(n) := \lim_{t \to \infty} P(X(t)=n) \) exists regardless of the initial condition, and for all \( n=(n_1,\ldots,n_R) \in \{0,1,\ldots\}^R \),

\[
\pi(n) = (1 - \rho)|n|! \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!}
\]

Result still true for arbitrary service time distr.!
Assume $\rho < 1$. We will use

$$\sum n! \prod_{r=1}^{R} x_r^{n_r} / n_r! = \left( \sum_{r=1}^{R} x_r \right)^n$$

- $P(X_i = n) = \ ?$

$$F_i(z) = \sum_{n \geq 0} P(X_i = n)z^n \quad \text{z-transform of } X_i$$

$$= \sum_n z^{n_i} \pi(n) = (1 - \rho) \sum_n z^{n_i} |n|! \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!}$$

$$= (1 - \rho) \sum_{n \geq 0} \sum_{n_1 + \ldots + n_R = n} |n|! \left( \prod_{r=1}^{i-1} \frac{\rho_r^{n_r}}{n_r!} \right) \left( \rho_i z \right)^{n_i} \left( \prod_{r=i+1}^{R} \frac{\rho_r^{n_r}}{n_r!} \right)$$

$$= (1 - \rho) \sum_{n \geq 0} (\rho_1 + \ldots + \rho_{i-1} + \rho_i z + \rho_{i+1} + \ldots + \rho_R)^n$$

$$= \frac{1 - \rho}{1 - \rho + \rho_i + \rho_i z} \quad \text{as } \rho = \sum_{r=1}^{R} \rho_r$$
Since \( 1/(1-az) = \sum_{n \geq 0} (az)^n \) we can write \( F_i(z) \) as

\[
F_i(z) = \frac{1 - \rho}{1 - \rho + \rho_i + \rho_i z} = \frac{1 - \rho}{1 - \rho + \rho_i} \cdot \frac{1}{1 - az} \quad \text{with} \quad a = \frac{\rho_i z}{1 - \rho + \rho_i}
\]

to get (inversion of z-transform)

\[
P(X_i = n) = \frac{1 - \rho}{1 - \rho + \rho_i} \left( \frac{\rho_i}{1 - \rho + \rho_i} \right)^n
\]

- \( N_i \) = expected number of class-i customers

\[
= (d/dz)F_i(z) \big|_{z=1} = \frac{\rho_i}{1 - \rho}
\]

(or can be obtained from \( P(X_i=n) \))
Performance metrics PS node (cont’)

- $T_i = \text{expected sojourn of class-i customers}$

$$T_i = \frac{N_i}{\lambda_i} = \frac{1}{\mu_i(1-\rho)} \quad \text{by Little}$$

- $W_i = \text{expected waiting time of class-i customers}$

$$W_i = T_i - \frac{1}{\mu_i} = \frac{\rho}{\mu_i(1-\rho)}$$
Performance metrics PS node (cont’)

- \( P(X_1+...+X_R=n) = \sum_{n_1+...+n_R=n} \pi(n) = (1-\rho) \sum_{n_1+...+n_R=n} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} \)

  total # cust. \( = (1-\rho) \left( \sum_{r=1}^{R} \rho_r \right)^n = (1-\rho) \rho^n \)

  Same as \( M/M/1 \) with traffic intensity \( \rho \)!

  In particular, \( E[X_1 + ... + X_R] = \rho/(1-\rho) \)

- Mean sojourn time of an arbitrary customer
  \( = \rho/(\lambda(1-\rho)) \) (Little) with \( \lambda := \sum_{1 \leq r \leq R} \lambda_r \)
Toward BCMP: Multiclass LIFO node

- Single server, infinite waiting room
- Any arriving customer preempts the one in service, if any, and starts being served at once
- R classes of customers: Assume for time being that service times of class-r customers is exponentially distributed with rate $\mu_r$
- Poisson arrival with rate $\lambda_r$ for class-r
- All service times and arrival processes independent

Set $\rho_r = \frac{\lambda_r}{\mu_r}$ and $\rho = \sum_{1 \leq r \leq R} \rho_r$
\[ \{X(t) := (X_1(t), \ldots, X_R(t)) \mid t \geq 0\}, \text{ with } X_r(t) = \# \text{ class-}r \text{ in node at time } t \]

\[ \Rightarrow \text{ not CTMC, missing class of customer in server} \]

\[ N(t) = \# \text{ customer at time } t \]

\[ \{I(t) := (I_1(t), \ldots, I_{N(t)}(t)) \mid t \geq 0\}, \text{ with } I_k(t) = \# \text{ class of customer in } k\text{th position at time } t, \text{ with position 1 being in server} \]

\[ \Rightarrow \text{This is a CTMC} \]

\[ \pi^*(r_1, \ldots, r_n) \left( \mu_{r_1} + \sum_{r=1}^{R} \lambda_r \right) = \pi^*(r_2, \ldots, r_n) \lambda_{r_1} + \sum_{r=1}^{R} \pi^*(r, r_1, \ldots, r_n) \mu_r, \quad (r_1, \ldots, r_n) \in \{1, \ldots, R\}^n \]

\[ \pi^*(0) \sum_{r=1}^{R} \lambda_r = \sum_{r=1}^{R} \pi^*(r) \mu_r \]
N(t) = \# customer at time \( t \)
\{I(t):=(I_1(t), \ldots, I_{N(t)}(t)) , t\geq 0\}, with \( I_k(t) = \# \) class of customer in \( k \)th position at time \( t \) when \( N(t)>0 \), with position 1 being in server, and \( X(t)=0 \) when \( N(t)=0 \)

**Balance Eqns (BE):**

\[
\pi^*(r_1,\ldots,r_n)\left(\mu_{r_n} + \sum_{r=1}^{R} \lambda_r\right) = \pi^*(r_1,\ldots,r_{n-1})\lambda_{r_n} + \sum_{r=1}^{R} \pi^*(r_1,\ldots,r_n,r)\mu_r, (r_1,\ldots,r_n) \in \{1,\ldots,R\}^n
\]

\[
\pi^*(0)\sum_{r=1}^{R} \lambda_r = \sum_{r=1}^{R} \pi^*(r)\mu_r
\]

**Solution:** \( \pi^*(0) = \frac{1}{G}, \pi^*(r_1,\ldots,r_n) = \frac{1}{G} \prod_{i=1}^{n} \rho_{r_i} \) for \( n=1,2,\ldots \), with \( \rho_r := \lambda_r/\mu_r \).
N(t) = # customer at time t
{I(t):=(I_1(t), ..., I_{N(t)}(t)) , t≥0}, with I_k(t) = # class of customer in kth position at time t, with position 1 being in server

\[ \pi^*(0) = \frac{1}{G}, \quad \pi^*(r_1, ..., r_n) = \frac{1}{G} \prod_{i=1}^{n} \rho_{r_i} \] for n=1,2,..., solves BE with

\[ G = \pi^*(0) + \sum_{n\geq1} \pi^*(r_1, ..., r_n) = 1 + \sum_{n\geq1} \prod_{i=1}^{n} \rho_{r_i} \]

\[ = 1 + \sum_{n\geq1} \left( \sum_{r=1}^{R} \rho_r \right)^n = \sum_{n\geq0} \left( \sum_{r=1}^{R} \rho_r \right)^n = (1 - \rho)^{-1} \] if \( \rho < 1 \)
Result (Limiting distr. \( I(t) := (I_1(t), \ldots, I_{N(t)}(t)) \), \( t \geq 0 \))

If \( \rho < 1 \) then

\[
\pi^*(r_1, \ldots, r_n) := \lim_{t \to \infty} P(I(t) = r_1, \ldots, r_n) = (1 - \rho) \prod_{i=1}^{n} \rho_{r_i} \text{ for } (r_1, \ldots, r_n) \in \{1, 2, \ldots\}^R
\]

\[
\pi^*(0) := \lim_{t \to \infty} P(N(t) = 0) = 1 - \rho
\]

Last step: going from limiting distribution of CTMC \( \{I(t), \ t \geq 0\} \) to limiting distribution of \( \{(X_1(t), \ldots, X_R(t)), \ t \geq 0\} \)
For \( n=(n_1,...,n_R) \in \{1,...\}^R \) with \(|n|! := n_1+...+n_R = n \geq 1\)

\[
\pi(n) := P(X_1 = n_1,...,X_R = n_R) = \sum_{(r_1,...,r_n)\in S(n_1,...,n_R)} \pi^*(r_1,...,r_n)
\]

where \( S(n_1,...,n_R) \) set of vectors in \( \{1,2,...,R\}^n \) with \( n_1 \) components equal to 1, ..., \( n_R \) components equal to \( R \)

\[
\pi(n) = (1-\rho) \sum_{(r_1,...,r_n)\in S(n_1,...,n_R)} \prod_{j=1}^{n} \rho_{r_j} = (1-\rho) \sum_{(r_1,...,r_n)\in S(n_1,...,n_R)} \rho_1^{n_1} ... \rho_R^{n_R}
\]

\[
= (1-\rho) \left( \prod_{r=1}^{R} \rho_r \right) \sum_{(r_1,...,r_n)\in S(n_1,...,n_R)} 1 = (1-\rho) \left( \prod_{r=1}^{R} \rho_r \right) \frac{n!}{n_1!...n_R!}
\]

\( \pi(O):=P(\text{queue empty})=\pi^*(O) \)
Toward BCMP: Multiclass LIFO node (con’t)

Result (Limiting distribution multiclass LIFO node)

If $\rho < 1$ (stability) with $\rho_r := \lambda_r / \mu_r$ for $r=1,...,R$ then

$$\pi(n) := \lim_{t \to \infty} P(X(t)=n) = (1 - \rho) |n|! \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r !}$$

for all $n=(n_1,...,n_R) \in \{0,1,...\}^R$

- Same result as for multiclass PS queue!

Therefore

- Same performance measures + holds for arbitrary service time distributions
Toward BCMP: Multiclass IS node

- Single server, infinite waiting room
- Infinitely many servers
- $R$ classes of customers: Assume for time being that service times of class-$r$ customers is exponentially distributed with rate $\mu_r$
- Poisson arrival with rate $\lambda_r$ for class-$r$
- All service times and arrival processes independent
\{X(t):=(X_1(t),...,X_R(t)), t \geq 0\}, with \(X_r(t) = \# \text{ class-} r \text{ in node at time } t\), is a \text{CTMC}

Transitions from state \(n=(n_1,\ldots,n_R)\):

- \(n \rightarrow n+e_r : \lambda_r\)
- \(n \rightarrow n-e_r : \mu_r n_r\)

\[
\pi(n) \sum_{r=1}^{R} (\lambda_r + \mu_r n_r) = \sum_{r=1}^{R} \pi(n-e_r) \lambda_r 1(n_r > 0) + \sum_{r=1}^{R} \pi(n+e_r) \mu_r (n_r + 1)
\]

Easy to check

\[
\prod_{r=1}^{R} \frac{(\lambda_r / \mu_r)^{n_r}}{n_r!}
\]

solves balance eqns
Normalizing constant

\[ \pi(n) = \frac{1}{G} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} \] 

with

\[ G := \sum \prod_{n_r \geq 0} \frac{\rho_r^{n_r}}{n_r!} = \prod_{r=1}^{R} \left( \sum_{n_r \geq 0} \frac{\rho_r^{n_r}}{n_r!} \right) = e^{\sum_{r=1}^{R} \rho_r} \]

so that

\[ \pi(n) = e^{-\sum_{r=1}^{R} \rho_r} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} \]
**Toward BCMP: Multiclass IS node (con’t)**

**Result** (Limiting distribution multiclass LIFO node)

\[
\pi(n) := \lim_{t \to \infty} P(X(t)=n) = e^{-\sum_{r=1}^{R} \rho_r} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} \quad \text{for all } n=(n_1, \ldots, n_R) \in \{0,1,\ldots\}^R \text{ with } \rho_r := \frac{\lambda_r}{\mu_r} \text{ for } r=1,\ldots,R
\]

- Holds for arbitrary service time distributions
- Always stable
Performance measures IS node

- \( P(X_1 + \ldots + X_R = n) = \sum_{n_1+\ldots+n_R=n} \pi(n) = e^{-\sum_{r=1}^{R} \rho_r} \sum_{n_1+\ldots+n_R=n} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} \)

  total \# cust.

  \[ = e^{-\sum_{r=1}^{R} \rho_r} \frac{1}{n!} \left( \sum_{r=1}^{R} \rho_r \right)^n \]

In particular, \( E[X_1 + \ldots + X_R] = e^{-\sum_{r=1}^{R} \rho_r} \sum_{n \geq 1} \frac{n}{n!} \left( \sum_{r=1}^{R} \rho_r \right)^n = \sum_{r=1}^{R} \rho_r \)

- Mean sojourn time of an arbitrary customer
  \[ = \frac{\sum_{r=1}^{R} \rho_r}{\sum_{r=1}^{R} \lambda_r} \] (Little)
Open BCMP network

- K nodes, R classes of customers
- 4 types of nodes: FIFO, IS, LIFO, PS
- Poisson exogeneous arrivals at node $i$, rate $\lambda_{i,r}^0 \geq 0$ with at least one rate non-zero (open network)
- Arbitrary service times distribution with mean $1/\mu_{i,r}$ for class-$r$ cust. at node $i \in \{IS, LIFO, PS\}$
- Exponentially distributed service times with mean $1/\mu_i$ for class-$r$ customers at node $i \in \text{FIFO}$
- With prob. $p_{i,r;j,s}$ a class-$r$ customer leaving node $i$ enters node $j$ as a class-$s$ customer
- All above rvs are mutually independent
Result (Limiting distribution open BCMP network)

If \((\lambda_{i,r}, i=1,...,K, j=1,...,R)\) is the unique positive solution of the traffic eqns

\[
\lambda_{i,r} = \lambda_{i,r}^0 + \sum_{1 \leq j \leq K, j \neq i, 1 \leq s \leq R} \lambda_{j,s} p_{j,s,i,r} \text{ for } i=1,...,K, \ r=1,...,R
\]

and if \(\sum_{r=1}^{R} \rho_{i,r} < 1\) for \(i \in \{\text{FIFO, LIFO, PS}\}\) with

\[
\rho_{i,r} := \begin{cases} 
\frac{\lambda_{i,r}}{\mu_{i,r}} & \text{for } i \in \text{IS, LIFO, PS} \\
\frac{\lambda_{i,r}}{\mu_i} & \text{for } i \in \text{FIFO} 
\end{cases}
\]

then
\[ \pi(n) := \lim_{t \to \infty} P(X(t)=n) = \prod_{i \in \text{FIFO}} f_i(n_i) \prod_{i \in \text{PS, LIFO}} f_i(n_i) \prod_{i \in \text{IS}} f_i(n_i) \]

for \( n=(n_1, \ldots, n_K) \), \( n_i = (n_{i,1}, \ldots, n_{i,R}) \) for \( i=1, \ldots, K \), where

\[
f_i(n_i) = \begin{cases} 
\left(1 - \sum_{r=1}^{R} \rho_{i,r}\right) |n_i|! \prod_{r=1}^{R} \frac{\rho_{i,r}^{n_{i,r}}}{n_{i,r}!} & \text{if } i \in \text{FIFO, LIFO, PS} \\
-e^{\sum_{r=1}^{R} \rho_{i,r}} \prod_{r=1}^{R} \frac{\rho_{i,r}^{n_{i,r}}}{n_{i,r}!} & \text{if } i \in \text{IS} 
\end{cases}
\]

with \( \rho_{i,r} := \frac{\lambda_{i,r}}{\mu_{i,r}} \)
Performance measures

- $X_i = \# \text{ customers in node } i \text{ in steady-state}$

\[
P(X_i = n) = \begin{cases} 
(1 - \rho_i)\rho_i^n & \text{for } i \in \text{FIFO, LIFO, PS} \\
e^{-\rho_i} \frac{\rho_i^n}{n!} & \text{for } i \in \text{IS}
\end{cases}
\]

with $\rho_i := \sum_{r=1}^{R} \rho_{i,r}$

- $E[X_i] = \begin{cases} 
\frac{\rho_i}{1 - \rho_i} & \text{for } i \in \text{FIFO, LIFO, PS} \\
\rho_i & \text{for } i \in \text{IS}
\end{cases}$
Performance measures (cont')

- $T = \text{expected sojourn time of arbitrary customer}$

$$T = \frac{\sum_{i=1}^{K} E[X_i]}{\sum_{1 \leq i \leq K, 1 \leq r \leq R} \lambda_{i,r}^0} = \frac{\sum_{i \in \text{FIFO, LIFO, PS}} \frac{\rho_i}{1 - \rho_i}}{\sum_{1 \leq i \leq K, 1 \leq r \leq R} \lambda_{i,r}^0} + \sum_{i \in \text{IS}} \rho_i$$

by Little’s formula

So long for BCMP networks!