Fluid modeling (con’t)

COMPSCI 655
Lecture 19
Motivation

Often systems are too « big » to allow for exact or even approximate microscopic analysis.

Call for macroscopic analysis to capture main system features/parameters/performance.

A - Mean-field approach (Lecture 18)

B - Poisson-driven stochastic differential equation (Lecture 18)

C - Ad hoc fluid models
Assume fluid enters an infinite reservoir at constant rate $\lambda$ and that reservoir is drained at constant rate $\mu$

Let $C(t)$ be content of reservoir at time $t$

If $C(0) = 0$ then $C(t) = 0$ if $\lambda \leq \mu$

$\hspace{1.5cm} = (\lambda - \mu)t$ if $\lambda > \mu$

or $C(t) = \max(0, (\lambda - \mu)t$

In particular, $C(t) \to \infty$ at $t \to \infty$ when $\lambda > \mu$
Assume now fluid arrival rate is governed by a stochastic process \( \{Z(t)\}_{t \geq 0} \), taking values in \( \{1,2,\ldots,N\} \).

- When \( Z(t) = i \) arrival rate at time is \( \lambda_i \geq 0 \\

\[
\frac{dC(t)}{dt} = 0 \quad \text{if} \quad C(t) = 0 \quad \text{and} \quad \lambda_{Z(t)} \leq \mu \\
= \lambda_{Z(t)} - \mu \quad \text{otherwise}
\]

when reservoir is infinite, and

\[
\frac{dC(t)}{dt}=0 \quad \text{if} \quad C(t)=0 \quad \text{and} \quad \lambda_{Z(t)} \leq \mu \quad \text{or if} \quad C(t)=K \quad \text{and} \quad \lambda_{Z(t)} > \mu \\
= \lambda_{Z(t)} - \mu \quad \text{otherwise}
\]

when reservoir of finite size \( K \)
Example 1: \( \{Z(t)\}_{t \geq 0} \) two-state CTMC with generator

\[
Q = \begin{pmatrix}
-\alpha & \alpha \\
\beta & -\beta
\end{pmatrix}
\]
and \( \lambda_i = \lambda > 0 \) if \( i=1 \) and \( \lambda_i = 0 \) if \( i=2 \)

⇒ so-called « On-off fluid source »

Q: \( \lim_{t \to \infty} P(C(t) < x) = ? \)

We assume that the reservoir is infinite
Example 1 (cont'): \( \{Z(t)\}_{t \geq 0} \) two-state CTMC, generator

\[
Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}
\]

and \( \lambda_i = \lambda > 0 \) if \( i=1 \) and \( \lambda_i = 0 \) if \( i=2 \)

\[ \lim_{t \to \infty} \mathbb{P}(Z(t)=i) := \mathbb{P}(Z=i) = ? \quad i=1,2 \]

Solving \((\mathbb{P}(On), \mathbb{P}(Off))Q = 0, \mathbb{P}(On) + \mathbb{P}(Off) = 1\)

\[
\mathbb{P}(On) = \frac{\beta}{\alpha + \beta}, \quad \mathbb{P}(Off) = \frac{\alpha}{\alpha + \beta}
\]
Example 1: Single On-Off fluid source

- $X_i$: length $i$-th On-period, exp. rv rate $\alpha$
- $Y_i$: length $i$-th Off-period, exp. rv rate $\beta$
- All rvs $\{X_i, Y_i\}_i$ are mutually independent

Buffer drained at rate $\mu$
Q: \( \lim_{t \to \infty} P(C(t) \leq x) = ? \) \( F_i(t,x) = P(C(t) \leq x, Z(t) = i) \)

Assume \( \lambda > \mu \) (otherwise \( C(t) = 0 \) for all if \( C(0)=0 \))

\[
F_1(t+h,x) = P(C(t+h) \leq x, Z(t+h) = \text{On}) =
\]
\[
P(C(t) \leq x+\mu h, Z(t) = \text{Off}) \cdot P(\text{duration Off period} < h)
\]
\[
+ P(C(t) \leq x-(\lambda-\mu)h, Z(t) = \text{On}) \cdot P(\text{duration On period} > h) + o(h)
\]

\( \Rightarrow \) as net input in \((t,t+h)\) when source On is \((\lambda-\mu)h\)

\[
F_2(t,x+\mu h)\beta h + F_1(t,x+(\mu-\lambda)h)(1-\alpha h) + o(h)
\]

\[
F_1(t+h,x) - F_1(t,x) + F_1(t,x) - F_1(t,x-(\lambda-\mu)h)
\]
\[
= \beta h F_2(t,x+\mu h) - \alpha h F_1(t,x+(\mu-\lambda)h)
\]

\[
\frac{[F_1(t+h,x) - F_1(t,x)]}{h} + (\lambda-\mu)\frac{[F_1(t,x) - F_1(t,x-(\lambda-\mu)h)]}{(\lambda-\mu)h}
\]
\[
= \beta F_2(t,x+\mu h) - \alpha F_1(t,x+(\mu-\lambda)h)
\]

\( h \to 0: \quad \frac{dF_1(t,x)}{dt} + (\lambda-\mu)\frac{dF_1(t,x)}{dx} = \beta F_2(t,x) - \alpha F_1(t,x) \)
\[
\frac{dF_1(t,x)}{dt} + (\lambda - \mu)\frac{dF_1(t,x)}{dx} = \beta F_2(t,x) - \alpha F_1(t,x)
\]

Similarly
\[
F_2(t+h,x) = P(C(t+h) \leq x, Z(t) = \text{Off}) = \alpha h F_1(t,x) + (1-\beta h)F_2(t,x+h\mu) + o(h)
\]

\[
\Rightarrow \frac{dF_2(t,x)}{dt} - \mu \frac{dF_2(t,x)}{dx} = \alpha F_1(t,x) - \beta F_2(t,x)
\]
\[ \frac{dF_1(t,x)}{dt} + (\lambda - \mu) \frac{dF_1(t,x)}{dx} = \beta F_2(t,x) - \alpha F_1(t,x) \]
\[ \frac{dF_2(t,x)}{dt} - \mu \frac{dF_2(t,x)}{dx} = \alpha F_1(t,x) - \beta F_2(t,x) \]

At equilibrium (with \( \lim_{t \to \infty} F_i(t,x) := F_i(x) \))

\[ (\lambda - \mu) \frac{dF_1(x)}{dx} = \beta F_2(x) - \alpha F_1(x) \]  
\[ -\mu \frac{dF_2(x)}{dx} = \alpha F_1(x) - \beta F_2(x) \]  

(4)

Adding both sides \( \Rightarrow (\lambda - \mu) \frac{dF_1(x)}{dx} - \mu \frac{dF_2(x)}{dx} = 0 \)

so that \( (\lambda - \mu) F_1(x) - \mu F_2(x) = c_0 \) with \( c_0 \) a constant

(4) becomes

\[ \frac{dF_1(x)}{dx} = \left( \frac{\beta}{\mu} - \frac{\alpha}{\lambda - \mu} \right) F_1(x) - \frac{\beta c_0}{(\lambda - \mu)\mu} \]
\[ \frac{dF_1(x)}{dx} = aF_1(x) + b \]

with \( a = \frac{\beta}{\mu} - \frac{\alpha}{(\lambda - \mu)}, \) \( b = -\frac{\beta c_0}{\mu(\lambda - \mu)} \)

Solution: \( F_1(x) = c_1 e^{ax} - \frac{b}{a} \)

Stable if \( a < 0 \iff \frac{\beta}{\mu} < \frac{\alpha}{(\lambda - \mu)} \iff \frac{\beta \lambda}{(\alpha + \beta)} < \mu \)

Makes sense as \( P(On) = \frac{\beta}{(\alpha + \beta)} \)

\( c_0 = ? \)

Defined by \((\lambda - \mu)F_1(x) = \mu F_2(x) + c_0\) for all \(x \geq 0\)

\((\lambda - \mu)F_1(\infty) = \mu F_2(\infty) + c_0\)

\( c_0 = (\lambda - \mu)P(X = 1) - \mu P(X = 2) \)

\( = ((\lambda - \mu)\beta - \mu \alpha)/(\alpha + \beta) \)
\( F_1(x) = c_1 e^{ax} - b/a \) with \( a = \beta/\mu - \alpha/(\lambda-\mu) \),
\( b := -\beta c_0/(\mu(\lambda-\mu)) \) and \( c_0 = ((\lambda-\mu)\beta - \mu\alpha)/((\alpha+\beta)) \)

\( c_1 = ? \)

\( F_1(0) = 0 \) implies \( c_1 = b/a \) yielding \( F_1(x) = -b/a(1-e^{ax}) \)

Moreover \( F_1(\infty) = P(On) = \beta/(\alpha+\beta) \) gives \( b/a = -\beta/(\alpha+\beta) \)

In summary

\[
F_1(x) = \frac{\beta}{\alpha + \beta} \left( 1 - e^{\left( \frac{\beta - \alpha}{\mu(\lambda-\mu)} \right)x} \right)
\]

\( F_2(x) \) obtained from \((\lambda-\mu)F_1(x) = \mu F_2(x) + c_0\)

\[
F_2(x) = \frac{\alpha}{\alpha + \beta} - \frac{\beta(\lambda - \mu)}{(\alpha + \beta)\mu} e^{\left( \frac{\beta - \alpha}{\mu(\lambda-\mu)} \right)x}
\]
\[ F_1(x) = \frac{\beta}{\alpha + \beta} \left( 1 - e^{\left( \frac{\beta - \alpha}{\mu - \lambda - \mu} \right) x} \right) \]

\[ F_2(x) = \frac{\alpha}{\alpha + \beta} - \frac{\beta(\lambda - \mu)}{(\alpha + \beta)\mu} e^{\left( \frac{\beta - \alpha}{\mu - \lambda - \mu} \right) x} \]

**Q:** Average fluid level?

\[ \bar{C} = \int_0^\infty P(C > x) \, dx = \int_0^\infty (1 - (F_1(x) + F_2(x))) \, dx \]

\[ = \frac{\beta \lambda (\lambda - \mu)}{\alpha + \beta} \left( \frac{1}{\mu(\lambda + \beta) - \lambda \beta} \right) \]
Let us come back to

$$(\lambda-\mu)dF_1(x)/dx = \beta F_2(x) - \alpha F_1(x)$$

$$-\mu dF_2(x)/dx = \alpha F_1(x) - \beta F_2(x)$$

Equivalent to

$$R \frac{dF(x)}{dx} = Q^T F(x) \quad \text{with} \quad F(x) = (F_1(x), F_2(x))^T$$

$$R = \begin{pmatrix} \lambda - \mu & 0 \\ 0 & -\mu \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

Since $\lambda > \mu$ \quad $\frac{dF(x)}{dx} = R^{-1}Q^T F(x)$
Return to general case: \( \{Z(t)\}_{t \geq 0} \) is an N-state CTMC

d\( C(t) \)/dt = 0 if \( C(t) = 0 \) and \( r_{Z(t)} \leq 0 \) or if \( C(t) = K \) and \( r_{Z(t)} > 0 \)

= \( r_{Z(t)} \) otherwise

with \( K \) finite or infinite (here \( r_{Z(t)} := \lambda_{Z(t)} - \mu \))

**Nota:** \( r_i \) is the net input rate in state \( i \). It can be positive or negative or equal to zero

**Q:** \( \lim_{t \to \infty} P(C(t) \leq x) = ? \)
\[ F(x) = (F_1(x), \ldots, F_N(x))^T \]
with \[ F_i(x) = \lim_{t \to \infty} P(C(t) \leq x, Z(t) = i) \]
\[ R = \text{diag}(r_1, \ldots, r_N) \]
\[ Q = [q_{i,j}]_{1 \leq i,j \leq N} \text{ generator of CTMC } \{Z(t)\}_{t \geq 0} \]

Stability condition when reservoir is infinite is
\[ \sum_{1 \leq i \leq N} r_i p_i < 0 \]
with \( p = (p_1, \ldots, p_N) \) stat. distr. of \( \{Z(t)\}_{t \geq 0} \) (i.e. \( pQ = 0, p \cdot \mathbf{1} = 1 \))

Easy to show that \( RF'(x) = Q^T F(x) \)

If \( r_i \neq 0 \) then \( R^{-1} \) exists and \( F'(x) = R^{-1}Q^T F(x) \)

From now on assume \( r_i \neq 0 \) for all \( i = 1, 2, \ldots, N \)
\[ F'(x) = R^{-1}Q^T F(x) \]

- When eigenvalues of \( R^{-1}Q^T \) are simple (i.e. they are all different)

\[
F(x) = \sum_{j=1}^{N} c_j e^{\xi_j x} v^{(j)}
\]

with

- \((\xi_j, v^{(j)})\) eigenvalue-eigenvector pairs of \( R^{-1}Q^T \)
- \(c_1, ..., c_N\) constants found from boundary conditions (see next)
Check that \( \overline{F}(x) = \sum_{j=1}^{N} c_j e^{\xi_j x} \overline{v}^{(j)} \) solves \( \overline{F}'(x) = \mathbf{R}^{-1} \mathbf{Q}^T \overline{F}(x) \)

\[
\overline{F}'(x) = \sum_{j=1}^{N} c_j e^{\xi_j x} \overline{\xi}_j \overline{v}^{(j)}
\]

\[
= \sum_{j=1}^{N} c_j e^{\xi_j x} \mathbf{R}^{-1} \mathbf{Q}^T \overline{v}^{(j)}
\]

as \( \mathbf{R}^{-1} \mathbf{Q}^T \overline{v}^{(j)} = \xi_j \overline{v}^{(j)} \)

\[
= \mathbf{R}^{-1} \mathbf{Q}^T \overline{N} \sum_{j=1}^{N} c_j e^{\xi_j x} \overline{v}^{(j)}
\]

\[
= \mathbf{R}^{-1} \mathbf{Q}^T \overline{F}(x)
\]
Assume $K$ is infinite from now on

$N_+ = \{i : r_i > 0\}, \ N_- = \{i : r_i < 0\}$

Note $\mathbin{\bigcup} N_+ \cap N_- = \{1, \ldots, N\}$ as $r_i \neq 0$ for all $i$

A remarkable result:
Under stability condition $\sum_{1 \leq i \leq N} r_i p_i < 0$

- nb. of eigenvalues with negative real part is $|N_+|$
- nb. of eigenvalues with positive real part is $|N_-|-1$
- 1 eigenvalue is zero

Rewrite as

$$F(x) = c_1 v^{(1)} + \sum_{j=2}^{\mid N_+ \mid + 1} c_j e^{\xi_j x} v^{(j)} + \sum_{j=\mid N_+ \mid + 2}^{N} c_j e^{\xi_j x} v^{(j)}$$

$\text{Re}(\xi_j) < 0$ for $j = 2, \ldots, \mid N_+ \mid + 1$, $\text{Re}(\xi_j) > 0$ for $j = \mid N_+ \mid + 2, \ldots, N$
\[ F(x) = c_1 v^{(1)} + \sum_{j=2}^{\lvert N_+ \rvert + 1} c_j e^{\xi_j x} v^{(j)} + \sum_{j=\lvert N_+ \rvert + 2}^{N} c_j e^{\xi_j x} v^{(j)} \]

with

\[ \text{Re}(\xi_j) < 0 \text{ for } j = 2, \ldots, \lvert N_+ \rvert + 1, \text{ Re}(\xi_j) > 0 \text{ for } j = \lvert N_+ \rvert + 2, \ldots, N \]

One must have \( c_j = 0 \) for \( j = \lvert N_+ \rvert + 2, \ldots, N \) as otherwise \( F(\infty) \) unbounded which is impossible as \( F(\infty) = p \)

Therefore

\[ F(x) = c_1 v^{(1)} + \sum_{j=2}^{\lvert N_+ \rvert + 1} c_j e^{\xi_j x} v^{(j)} \]

with \( \text{Re}(\xi_j) < 0 \) for \( j = 2, \ldots, \lvert N_+ \rvert + 1 \)
Determining constants \( c_j \)'s when \( K \) is infinite

- \( F(\infty) = \underline{p} \) with \( \underline{p} = (p_1, \ldots, p_N) \) stationary distribution of CTMC modulating input rates

Gives \( c_1 \underline{v}^{(1)} = \underline{p} \) and

\[
F(x) = \underline{p} + \sum_{j=2}^{|N_+|+1} c_j e^{\xi_j x} \underline{v}^{(j)}
\]

- \( F_i(0) = 0 \) when \( r_i > 0 \) (buffer cannot be empty in steady-state if net input rate strictly positive)

\[ \rightarrow \text{there are exactly } N_+ \text{ states } i \text{ s.t. } F_i(0) = 0 \]

Yields linear system of \( N_+ \) equations

\[
A \underline{c} = \underline{b}^T \text{ with } \underline{c} = (c_j, j = 2, \ldots, |N_+|+1)^T
\]

\[
\underline{b} = (p_j, j \in N_+)^T
\]
$W = \text{Average amount of fluid}$

$$W = \int_0^\infty \left( 1 - \sum_{i=1}^{N} F_i(x) \right) dx$$
Example 2: The celebrated Anick et al. model

Superposition of $N$ independent, statistically identical on-off fluid sources, infinite buffer

- $X^1_n$: length $i$th On-period source $n$, exp. rate $\alpha$
- $Y^1_n$: length $i$th Off-period source $n$, exp. rate $\beta$
- All rvs \{X^m_n, Y^m_n\}_{m,n,i,j} are mutually independent
Example 2: The celebrated Anick et al. model (cont’)

- Each source has generator \( \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} \)
- For each source, arrival rate in state On is 1, arrival rate in state Off is 0
- In steady-state, each source in state On/Off with prob. \( \frac{\beta}{\alpha + \beta}, \frac{\alpha}{\alpha + \beta} \), resp.

Let \( N(t) = \# \) sources in state On at time \( t \)

\[ \pi_n := \lim_{t \to \infty} P(N(t)=n) = \binom{N}{n} \beta^n \alpha^{N-n} / (\alpha + \beta)^N \]
Example 2: The celebrated Anick et al. model (cont’)

\[ \pi_n = \binom{N}{n} \beta^n \alpha^{N-n} / (\alpha + \beta)^N \] prob. \( n \) sources On

- Expected number of sources in state On

\[ \sum_{n=0}^{N} n\pi_n = \frac{1}{(\alpha + \beta)^N} \sum_{n=0}^{N} \binom{N}{n} \beta^n \alpha^{N-n} = \frac{N\beta}{(\alpha + \beta)} \]

- System stable if \( \frac{N\beta}{(\alpha + \beta)} < r \)
Example 2: The celebrated Anick et al. model (cont')

Can be cast into previous framework, as superposition of N independent CTMCs is CTMC

\[ Z(t) = \text{# sources in state On at time } t \]

\( \{Z(t)\}_{t \geq 0} \) is a CTMC with non-zero transition rates

\[ i \rightarrow i+1 \text{ with rate } (N-i)\beta \]

\[ i \rightarrow i-1 \text{ with rate } i\alpha \]

Previous theory applies to CTMC \( \{(C(t),Z(t))\}_{t \geq 0} \) with \( C(t) \) amount of fluid in buffer at time \( t \)
Example 2: The celebrated Anick et al. model (cont’)

Previous theory applies to CTMC \{\{(C(t),Z(t))\}\}_{t \geq 0}
with \(C(t)\) amount of fluid in buffer at time \(t\)

Define:

\[
F_n(x) = \lim_{t \to \infty} P(C(t) < x, Z(t) = n)
\]

\[
F(x) = (F_1(x),...,F_N(x))^T
\]

We get

\[
\begin{align*}
F(x) &= \sum_{j=1}^{N} c_j e^{\xi_j x} \varphi^{(j)} \\
\text{with} & \\
(\xi_j, \varphi^{(j)}) & \text{eigenvalue-eigenvector pairs of } R^{-1}Q^T
\end{align*}
\]
Example 2: The celebrated Anick et al. model (cont’)

\[ F(x) = \sum_{j=1}^{N} c_j e^{s_j x} \nu^{(j)} \] with \( R^{-1} Q^T \nu^{(j)} = \xi_j \nu^{(j)} \), \( j=1,...,N \)

Here \( R = \text{diag}(-r, 1-r, \ldots, N-r) \) (take \( r \) not integer) and

\[
Q = \begin{pmatrix}
-\beta N & \beta N \\
\alpha & -\left(\alpha + (N-1)\beta\right) & (N-1)\beta \\
2\alpha & -\left(2\alpha + (N-2)\beta\right) & (N-2)\beta \\
& \ddots & \ddots \\
& & (N-1)\alpha & -\left((N-1)\alpha + \beta\right) & \beta \\
& & (N\alpha) & -N\alpha
\end{pmatrix}
\]
Example 2: The celebrated Anick et al. model (cont’)

Q: Why is this model famous?

A: Because all quantities in

\[ F(x) = \sum_{j=1}^{N} c_j e^{\zeta_j x} v^{(j)} \]

can be obtained in explicit form!

See:

Last words

- Anick et al. model can be extended to independent non-statistical identical On-Off sources 😊

- No product-form like-result around the corner for network of fluid queues 😞