Fluid modeling (con’t)

COMPSCI 655
Lecture 19
Motivation

Often systems are too « big » to allow for exact or even approximate microscopic analysis

Call for macroscopic analysis to capture main system features/parameters/performance

A - Mean-field approach (Lecture 18)
B - Poisson-driven stochastic differential equation (Lecture 18)
C - Ad hoc fluid models
Assume fluid enters an infinite reservoir at constant rate $\lambda$ and that reservoir is drained at constant rate $\mu$.

Let $C(t)$ be content of reservoir at time $t$.

If $C(0) = 0$ then $C(t) = 0$ if $\lambda \leq \mu$

$$= (\lambda - \mu)t \text{ if } \lambda > \mu$$

or $C(t) = \max(0, (\lambda - \mu))t$

In particular, $C(t) \to \infty$ at $t \to \infty$ when $\lambda > \mu$. 
Assume now fluid arrival rate is governed by a stochastic process \( \{Z(t)\}_{t \geq 0} \), taking values in \( \{1,2,\ldots,N\} \)

- When \( Z(t) = i \) arrival rate at time is \( \lambda_i \geq 0 \)

\[
dC(t)/dt = 0 \quad \text{if } C(t) = 0 \text{ and } \lambda_{Z(t)} \leq \mu \\
= \lambda_{Z(t)} - \mu \quad \text{otherwise}
\]

when reservoir is infinite, and

\[
dC(t)/dt=0 \text{ if } C(t)=0 \text{ and } \lambda_{Z(t)} \leq \mu \text{ or if } C(t)=K \text{ and } \lambda_{Z(t)} > \mu \\
= \lambda_{Z(t)} - \mu \quad \text{otherwise}
\]

when reservoir of finite size \( K \)
Example 1: \( \{Z(t)\}_{t \geq 0} \) two-state CTMC with generator
\[
\begin{pmatrix}
\ -\alpha & \alpha \\
\beta & -\beta
\end{pmatrix}
\]
and \( \lambda_i = \lambda > 0 \) if \( i=1 \) and \( \lambda_i = 0 \) if \( i=2 \)

\( \Rightarrow \) so-called « On-off fluid source »

Q: \( \lim_{t \to \infty} P(C(t) < x) = ? \)

We assume that the reservoir is infinite
Example 1 (cont'): \( \{Z(t)\}_{t \geq 0} \) two-state CTMC, generator

\[
Q = \begin{pmatrix}
-\alpha & \alpha \\
\beta & -\beta
\end{pmatrix}
\]

and \( \lambda_i = \lambda > 0 \) if \( i=1 \) and \( \lambda_i = 0 \) if \( i=2 \)

\[\lim_{t \to \infty} P(Z(t)=i) := P(Z=i) = ? \quad i=1,2\]

Solving \( (P(On), P(Off))Q = 0, P(On) + P(Off) = 1 \)

gives

\[
P(On) = \frac{\beta}{\alpha + \beta}, \quad P(Off) = \frac{\alpha}{\alpha + \beta}
\]
Example 1: Single On-Off fluid source

- $X_i$: length $i$-th On-period, exp. rv rate $\alpha$
- $Y_i$: length $i$-th Off-period, exp. rv rate $\beta$
- All rvs $\{X_i, Y_i\}_i$ are mutually independent
Q: \[ \lim_{t \to \infty} P(C(t) \leq x) = ? \quad x > 0 \quad F_i(t, x) := P(C(t) \leq x, Z(t) = i) \]

Assume \( \lambda > \mu \) (otherwise \( C(t) = 0 \) for all if \( C(0) = 0 \))

\[ F_1(t+h, x) = P(C(t+h) \leq x, Z(t+h) = \text{On}) = \]
\[ P(C(t+h) \leq x, Z(t+h) = \text{On}, Z(t) = \text{OFF}) + P(C(t+h) \leq x, Z(t+h) = \text{On}, Z(t) = \text{ON}) \]

First consider \( P(C(t+h) \leq x, Z(t+h) = \text{On}, Z(t) = \text{OFF}) = * \)

\[ *= \int_{y=t}^{t+h} P(C(t+h) \leq x, Z(t+h) = \text{On}, Z(t) = \text{On}| \text{On} \to \text{Off at time } t+y) \cdot dP(\text{On} \to \text{Off at time } t+y) + o(h) \]

\[ = \int_{y=t}^{t+h} P(C(t) \leq h(\lambda - \mu) - \lambda y, Z(t) = \text{On}) \beta e^{-\beta y} dy + o(h) \]

as net input in \( (t, t+y) \) is \(-\mu y\) when source \text{Off} in \( (t, t+y)\), and net input in \( (t+y, t+h) \) is \((\lambda - \mu)(h-y)\) when source \text{On} in \( (t, t+y)\)

Total net input in \( (t, t+h) = -\mu y + (\lambda - \mu)(h-y) = h(\lambda - \mu) - \lambda y \)

\( o(h) \) corresponds to probability that source state changes at least twice in \( (t, t+h) \).
Therefore

\[ P(C(t+h) \leq x, Z(t+h) = \text{On}, Z(t) = \text{OFF}) = \]

\[ = \int_{y=t}^{t+h} P(C(t) \leq x + h(\lambda-\mu)-\lambda y, Z(t) = \text{On}) \beta e^{-\beta y} dy + o(h) \]

\[ = \beta h h^{-1} \int_{y=t}^{t+h} F_2(t, x + h(\lambda-\mu)-\lambda y) e^{-\beta y} dy \]

\[ = \beta h F_2(t, x) + o(h) \]

by Mean Value Theorem
Consider now \( P(C(t+h) \leq x, Z(t+h) = On, Z(t) = On) = ** \)

\[
** = P(C(t+h) \leq x, Z(t+h) = On, Z(t) = On)
\]

\[
= P(C(t) \leq x-(\lambda-\mu)h, Z(t) = On).P(\text{duration On period } >h) + o(h)
\]

as net input in \((t, t+h)\) when source On is \((\lambda-\mu)h\)

\[
= F_1(t,x+(\mu-\lambda)h)(1-\alpha h) + o(h)
\]

\(o(h)\) corresponds to probability that source state changes at least twice in \((t, t+h)\).

In summary

\[
F_1(t+h,x) - F_1(t,x) + F_1(t,x) - F_1(t,x-(\lambda-\mu)h)
\]

\[
= \beta hF_2(t,x) - \alpha hF_1(t,x+(\mu-\lambda)h) + o(h)
\]
\[ F_1(t+h,x) - F_1(t,x) + F_1(t,x) - F_1(t,x-(\lambda-\mu)h) = \beta h F_2(t,x) - \alpha h F_1(t,x+(\mu-\lambda)h) + o(h) \]

Dividing by \( h \), then letting \( h \to 0 \) gives

\[ \frac{dF_1(t,x)}{dt} + (\lambda-\mu)\frac{dF_1(t,x)}{dx} = \beta F_2(t,x) - \alpha F_1(t,x) \]

Similarly

\[ F_2(t+h,x) = P(C(t+h) \leq x, Z(t) = \text{Off}) = \alpha h F_1(t,x) + (1-\beta h)F_2(t,x+h\mu)+o(h) \]

\[ \to \frac{dF_2(t,x)}{dt} - \mu \frac{dF_2(t,x)}{dx} = \alpha F_1(t,x)-\beta F_2(t,x) \]
\[
\frac{dF_1(t,x)}{dt} + (\lambda - \mu) \frac{dF_1(t,x)}{dx} = \beta F_2(t,x) - \alpha F_1(t,x)
\]
\[
\frac{dF_2(t,x)}{dt} - \mu \frac{dF_2(t,x)}{dx} = \alpha F_1(t,x) - \beta F_2(t,x)
\]

At equilibrium (with \( \lim_{t \to \infty} F_i(t,x) := F_i(x) \))

\[
(\lambda - \mu) \frac{dF_1(x)}{dx} = \beta F_2(x) - \alpha F_1(x)
\]
\[
- \mu \frac{dF_2(x)}{dx} = \alpha F_1(x) - \beta F_2(x)
\]

Adding both sides \( \Rightarrow (\lambda - \mu) \frac{dF_1(x)}{dx} - \mu \frac{dF_2(x)}{dx} = 0 \)

so that \( (\lambda - \mu) F_1(x) - \mu F_2(x) = c_0 \) with \( c_0 \) a constant

(4) becomes

\[
\frac{dF_1(x)}{dx} = \left( \frac{\beta}{\mu} - \frac{\alpha}{\lambda - \mu} \right) F_1(x) - \frac{\beta c_0}{(\lambda - \mu)\mu}
\]
\[
\frac{dF_1(x)}{dx} = aF_1(x) + b
\]

with \( a = \frac{\beta}{\mu} - \frac{\alpha}{(\lambda - \mu)} \), \( b = -\frac{\beta c_0}{\mu(\lambda - \mu)} \)

**Solution:** \( F_1(x) = c_1 e^{ax} - \frac{b}{a} \)

Stable if \( a < 0 \iff \frac{\beta}{\mu} < \frac{\alpha}{(\lambda - \mu)} \iff \frac{\beta \lambda}{(\alpha + \beta)} < \mu \)

Makes sense as \( P(\text{On}) = \frac{\beta}{(\alpha + \beta)} \)

\( c_0 = ? \)

Defined by \((\lambda - \mu)F_1(x) = \mu F_2(x) + c_0\) for all \( x \geq 0 \)

\((\lambda - \mu)F_1(\infty) = \mu F_2(\infty) + c_0\)

\( c_0 = (\lambda - \mu)P(X = 1) - \mu P(X = 2) \)

\( = \frac{((\lambda - \mu)\beta - \mu \alpha)}{(\alpha + \beta)} \)
\[ F_1(x) = c_1 e^{ax} - \frac{b}{a} \quad \text{with} \quad a = \frac{\beta}{\mu} - \frac{\alpha}{(\lambda-\mu)}, \]

\[ b := -\beta c_0/(\mu(\lambda-\mu)) \quad \text{and} \quad c_0 = ((\lambda-\mu)\beta - \mu\alpha)/(\alpha+\beta) \]

\[ c_1 = ? \]

\[ F_1(0) = 0 \implies c_1 = \frac{b}{a} \text{ yielding } F_1(x) = -\frac{b}{a}(1-e^{ax}) \]

Moreover \( F_1(\infty) = P(On) = \frac{\beta}{(\alpha+\beta)} \) gives \( \frac{b}{a} = -\frac{\beta}{(\alpha+\beta)} \)

In summary

\[
F_1(x) = \frac{\beta}{\alpha + \beta} \left( 1 - e^{\left( \frac{\beta}{\mu} - \frac{\alpha}{\lambda - \mu} \right) x} \right)
\]

\[ F_2(x) \text{ obtained from } (\lambda-\mu)F_1(x) = \mu F_2(x) + c_0 \]

\[
F_2(x) = \frac{\alpha}{\alpha + \beta} - \frac{\beta(\lambda - \mu)}{(\alpha + \beta)\mu} e^{\left( \frac{\beta}{\mu} - \frac{\alpha}{\lambda - \mu} \right) x}
\]
\[ F_1(x) = \frac{\beta}{\alpha + \beta} \left( 1 - e^{\left(\frac{\beta - \alpha}{\mu - \lambda - \mu}\right)x} \right) \]

\[ F_2(x) = \frac{\alpha}{\alpha + \beta} - \frac{\beta(\lambda - \mu)}{(\alpha + \beta)\mu} e^{\left(\frac{\beta - \alpha}{\mu - \lambda - \mu}\right)x} \]

**Q:** Average fluid level?

\[
\bar{C} = \int_0^\infty P(C > x) \, dx = \int_0^\infty (1 - (F_1(x) + F_2(x))) \, dx
\]

\[
\bar{C} = \frac{\beta \lambda (\lambda - \mu)}{\alpha + \beta} \left( \frac{1}{\mu(\lambda + \beta) - \lambda \beta} \right)
\]
Let us come back to
\[(\lambda - \mu)\frac{dF_1(x)}{dx} = \beta F_2(x) - \alpha F_1(x)\]
\[-\mu \frac{dF_2(x)}{dx} = \alpha F_1(x) - \beta F_2(x)\]

Equivalent to
\[R \frac{dF(x)}{dx} = Q^T F(x)\]
with \[F(x) = (F_1(x), F_2(x))^T\]

\[R = \begin{pmatrix} \lambda - \mu & 0 \\ 0 & -\mu \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}\]

Since \[\lambda > \mu\]
\[\frac{dF(x)}{dx} = R^{-1}Q^T F(x)\]
Return to general case: \( \{Z(t)\}_{t \geq 0} \) is an N-state CTMC

\[
dC(t)/dt = 0 \text{ if } C(t) = 0 \text{ and } r_{Z(t)} \leq 0 \text{ or if } C(t) = K \text{ and } r_{Z(t)} > 0
\]

\[
= r_{Z(t)} \text{ otherwise}
\]

with \( K \) finite or infinite (here \( r_{Z(t)} := \lambda_{Z(t)} - \mu \))

**Nota:** \( r_i \) is the net input rate in state \( i \). It can be positive or negative or equal to zero

**Q:** \( \lim_{t \to \infty} P(C(t) \leq x) = ? \)
\[ \mathbf{F}(x) = (F_1(x), ..., F_N(x))^T \]
with \( F_i(x) = \lim_{t \to \infty} P(C(t) \leq x, Z(t) = i) \)

\[ \mathbf{R} = \text{diag}(r_1, ..., r_N) \]

\[ \mathbf{Q} = [q_{i,j}]_{1 \leq i,j \leq N} \text{ generator of CTMC } \{Z(t)\}_{t \geq 0} \]

Stability condition when reservoir is infinite is

\[ \sum_{1 \leq i \leq N} r_i p_i < 0 \]

with \( p = (p_1, ..., p_N) \) stat. distr. of \( \{Z(t)\}_{t \geq 0} \) (i.e. \( p\mathbf{Q} = 0 \), \( p.1 = 1 \))

Easy to show that \( \mathbf{R}\mathbf{F}'(x) = \mathbf{Q}^T \mathbf{F}(x) \)

If \( r_i \neq 0 \) then \( \mathbf{R}^{-1} \) exists and \( \mathbf{F}'(x) = \mathbf{R}^{-1}\mathbf{Q}^T \mathbf{F}(x) \)

From now on assume \( r_i \neq 0 \) for all \( i = 1, 2, ..., N \)
\[ F'(x) = R^{-1}Q^T F(x) \]

- When eigenvalues of \( R^{-1}Q^T \) are simple (i.e. they are all different)

\[
F(x) = \sum_{j=1}^{N} c_j e^{\xi_j x} \lambda^{(j)}
\]

with
- \((\xi_j, \lambda^{(j)})\) eigenvalue-eigenvector pairs of \( R^{-1}Q^T \)
- \(c_1, \ldots, c_N\) constants found from boundary conditions (see next)
Check that

$$F'(x) = \sum_{j=1}^{N} c_j e^{\xi_j x} \underline{v}(j)$$

solves

$$F'(x) = R^{-1} Q^T F(x)$$

$$F'(x) = \sum_{j=1}^{N} c_j e^{\xi_j x} \xi_j \underline{v}(j)$$

$$= \sum_{j=1}^{N} c_j e^{\xi_j x} R^{-1} Q^T \underline{v}(j)$$

as

$$R^{-1} Q^T \underline{v}(j) = \xi_j \underline{v}(j)$$

$$= R^{-1} Q^T \sum_{j=1}^{N} c_j e^{\xi_j x} \underline{v}(j)$$

$$= R^{-1} Q^T F(x)$$
Assume $K$ is infinite from now on

$N_+ = \{i : r_i > 0\}$, $N_- = \{i : r_i < 0\}$

Note that $N_+ \cup N_- = \{1, ..., N\}$ as $r_i \neq 0$ for all $i$.

A remarkable result:

Under stability condition $\sum_{1 \leq i \leq N} r_i p_i < 0$

- nb. of eigenvalues with negative real part is $|N_+|$
- nb. of eigenvalues with positive real part is $|N_-|-1$
- 1 eigenvalue is zero

Rewrite as

$$F(x) = c_1 v^{(1)} + \sum_{j=2}^{\lfloor N+1 \rfloor} c_j e^{\xi_j x} v^{(j)} + \sum_{j=\lfloor N_+ \rfloor+2}^{N} c_j e^{\xi_j x} v^{(j)}$$

with $\text{Re}(\xi_j) < 0$ for $j=2, ..., \lfloor N_+ \rfloor+1$, $\text{Re}(\xi_j) > 0$ for $j=\lfloor N_+ \rfloor+2, ..., N$
\[ F(x) = c_1 v^{(1)} + \sum_{j=2}^{\left|N_+\right|+1} c_j e^{\xi_j x} v^{(j)} + \sum_{j=\left|N_+\right|+2}^{N} c_j e^{\xi_j x} v^{(j)} \]

with

\( \Re(\xi_j) < 0 \) for \( j = 2, \ldots, \left|N_+\right|+1 \), \( \Re(\xi_j) > 0 \) for \( j = \left|N_+\right|+2, \ldots, N \)

One must have \( c_j = 0 \) for \( j = \left|N_+\right|+2, \ldots, N \) as otherwise \( F(\infty) \) unbounded which is impossible as \( F(\infty) = p \)

Therefore

\[ F(x) = c_1 v^{(1)} + \sum_{j=2}^{\left|N_+\right|+1} c_j e^{\xi_j x} v^{(j)} \]

with \( \Re(\xi_j) < 0 \) for \( j = 2, \ldots, \left|N_+\right|+1 \)
Determining constants $c_j$'s when $K$ is infinite

- $\mathbb{F}(\infty) = \mathbb{p}$ with $\mathbb{p} = (p_1, ..., p_N)$ stationary distribution of CTMC modulating input rates

Gives $c_1 \mathbb{v}^{(1)} = \mathbb{p}$ and

$$\mathbb{F}(x) = \mathbb{p} + \sum_{j=2}^{|\mathbb{N}_+|+1} c_j e^{\xi_j x} \mathbb{v}^{(j)}$$

- $\mathbb{F}_i(0) = 0$ when $r_i > 0$ (buffer cannot be empty in steady-state if net input rate strictly positive) → there are exactly $\mathbb{N}_+$ states $i$ s.t. $\mathbb{F}_i(0) = 0$

Yields linear system of $\mathbb{N}_+$ equations

$$\mathbb{A} \mathbb{c} = \mathbb{b}^T$$ with $\mathbb{c} = (c_j, j = 2, ..., |\mathbb{N}_+|+1)^T$

$$\mathbb{b} = (p_j, j \in \mathbb{N}_+)^T$$
$$W = \text{Average amount of fluid}$$

$$W = \int_{0}^{\infty} \left( 1 - \sum_{i=1}^{N} F_{i}(x) \right) dx$$
Example 2: The celebrated Anick et al. model

Superposition of $N$ independent, statistically identical on-off fluid sources, infinite buffer

- $X^{1}_i$: length $i$th On-period source $n$, exp. rate $\alpha$
- $Y^{1}_j$: length $j$th Off-period source $n$, exp. rate $\beta$
- All rvs $\{X^{n}_i, Y^{m}_j\}_{m,n,i,j}$ are mutually independent
Example 2: The celebrated Anick et al. model (cont’)

- Each source has generator \[
\begin{pmatrix}
-\alpha & \alpha \\
\beta & -\beta
\end{pmatrix}
\]
- For each source, arrival rate in state On is 1, arrival rate in state Off is 0
- In steady-state, each source in state On/Off with prob. \(\frac{\beta}{(\alpha + \beta)}, \frac{\alpha}{(\alpha + \beta)}\), resp.

Let \(N(t) = \#\) sources in state On at time \(t\)

\[\pi_n := \lim_{t \to \infty} P(N(t)=n) = \binom{N}{n} \beta^n \alpha^{N-n} / (\alpha + \beta)^N\]
Example 2: The celebrated Anick et al. model (cont’)

\[ \pi_n = \binom{N}{n} \beta^n \alpha^{N-n} / (\alpha + \beta)^N \]  prob. n sources On

- Expected number of sources in state On

\[ \sum_{n=0}^{N} n \pi_n = \frac{1}{(\alpha + \beta)^N} \sum_{n=0}^{N} n \binom{N}{n} \beta^n \alpha^{N-n} = \frac{N \beta}{(\alpha + \beta)} \]

- System stable if \( \frac{N \beta}{(\alpha + \beta)} < r \)
Example 2: The celebrated Anick et al. model (cont’)

Can be cast into previous framework, as superposition of N independent CTMCs is CTMC

\[ Z(t) = \# \text{ sources in state On at time } t \]

\{Z(t)\}_{t \geq 0} \text{ is a CTMC with non-zero transition rates}

\begin{align*}
  i \to i+1 & \text{ with rate } (N-i)\beta \\
  i \to i-1 & \text{ with rate } i\alpha
\end{align*}

Previous theory applies to CTMC \{(C(t),Z(t))\}_{t \geq 0}

with \( C(t) \) amount of fluid in buffer at time \( t \)
Example 2: The celebrated Anick et al. model (cont’)

Previous theory applies to CTMC \{((C(t),Z(t)))\}_{t \geq 0}
with $C(t)$ amount of fluid in buffer at time $t$

Define:

$$F_n(x) = \lim_{t \to \infty} P(C(t) < x, Z(t) = n)$$

$$F(x) = (F_1(x), \ldots, F_N(x))^T$$

We get:

$$F(x) = \sum_{j=1}^{N} c_j e^{\xi_j x} \nu^{(j)}$$

with

- $(\xi_j, \nu^{(j)})$ eigenvalue-eigenvector pairs of $R^{-1}Q^T$
Example 2: The celebrated Anick et al. model (cont’)

\[ F(x) = \sum_{j=1}^{N} c_j e^{s_j x} v^{(j)} \quad \text{with} \quad R^{-1}Q^T v^{(j)} = \xi_j v^{(j)}, \ j=1, \ldots, N \]

Here \( R = \text{diag}(-r, 1-r, \ldots, N-r) \) (take \( r \) \text{ not} integer) and

\[
Q = \begin{pmatrix}
-\beta N & \beta N \\
\alpha & -(\alpha + (N-1)\beta) & (N-1)\beta \\
2\alpha & -(2\alpha + (N-2)\beta) & (N-2)\beta \\
& \ddots & \ddots \\
(N-1)\alpha & -((N-1)\alpha + \beta) & \beta \\
N\alpha & -N\alpha 
\end{pmatrix}
\]
Example 2: The celebrated Anick et al. model (cont’)

Q: Why is this model famous?

A: Because all quantities in

\[ F(x) = \sum_{j=1}^{N} c_j e^{\varsigma_j x} \nu^{(j)} \]

can be obtained in explicit form!

See:

Last words

- Anick et al. model can be extended to independent non-statistical identical On-Off sources 😊

- No product-form like-result around the corner for network of fluid queues 😞