NETWORKS OF QUEUES WITH CUSTOMERS OF DIFFERENT TYPES

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Abstract

The behaviour in equilibrium of networks of queues in which customers may be of different types is studied. The type of a customer is allowed to influence his choice of path through the network and, under certain conditions, his service time distribution at each queue. The model assumed will usually cause each service time distribution to be of a form related to the negative exponential distribution.

Theorems 1 and 2 establish the equilibrium distribution for the basic model in the closed and open cases; in the open case the individual queues are independent in equilibrium. In Section 4 similar results are obtained for other models, models which include processes better described as networks of colonies or as networks of stacks. In Section 5 the effect of time reversal upon certain processes is used to obtain further information about the equilibrium behaviour of those processes.

1. Introduction

We shall be concerned with systems in which there are $J$ queues and $I$ types of customer. If queue $j$ ($j = 1, 2, \ldots, J$) contains $n_j$ customers then we can describe the queue by $c_j = (c_j(1), c_j(2), \ldots, c_j(n_j))$ where $c_j(l) \in \{1, 2, \ldots, I\}$ is the type of the customer in position $l$ in the queue. If queue $j$ is empty then define $c_j = e$. The state of the system is $C = (c_1, c_2, \ldots, c_J)$ and we suppose that $C = C(t)$ is a Markov process in continuous time, $t$, with $q(C, D)$ denoting the transition rate from $C$ to $D$. (We assume that the transition rates determine uniquely the Markov process. This point is considered by Kendall [6] and Kendall and Reuter [5].) We shall be concerned with processes $C$ which evolve within an irreducible class $\mathcal{I}$; in cases where there may be more than one irreducible class we shall assume that we know which class the process evolves within and this class will be $\mathcal{I}$.

If customers cannot enter or leave the system of queues, but can only move between queues, then the system is closed; otherwise the system is open. In a closed system the total number of customers of each type is constant over states in $\mathcal{I}$ and hence $\mathcal{I}$ is finite. In an open system $\mathcal{I}$ may be countably infinite.

Received in revised form 21 January 1975.

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We shall attempt to find the equilibrium distribution on $\mathcal{S}$: that is positive numbers $P(C)$ ($C \in \mathcal{S}$) which satisfy

\begin{equation}
P(C) \sum_{D \in \mathcal{S}} q(C, D) = \sum_{D \in \mathcal{S}} P(D)q(D, C) \quad C \in \mathcal{S}
\end{equation}

\begin{equation}
\sum_{C \in \mathcal{S}} P(C) = 1.
\end{equation}

When $\mathcal{S}$ is finite these equations have one and only one solution. When $\mathcal{S}$ is infinite the position is more complicated, in that these equations may not have a solution, but if we can find a solution then it is unique ([5], [6]).

If all the customers are of the same type then the state of the system can be taken as $(n_1, n_2, \ldots, n_J)$; the order within a queue is no longer needed. Results for processes of this kind, which are often called Markov population processes, have been given by Jackson [3], Whittle [10], [11] and Kingman [7]. In such processes the customers in a particular queue are homogeneous. This paper can be regarded as a generalization of such processes to allow the customers in a particular queue to be heterogeneous. As an example consider a situation in which a customer enters the system and visits each queue exactly once, but in a random order, before leaving the system. We cannot model this situation as a Markov population process since it is not sufficient to take as the state of the system the number of customers in each queue. We must also include some indication of the queues which each customer has yet to visit, and this will cause the customers in a particular queue to be heterogeneous. We shall see that the situation can be modelled with the processes of this paper, simply by associating a type of customer with each possible ordering of the queues of the system.

Collings [2] has considered a single queue in which customers may be of different types, the type of a customer determining his service time distribution.

For ease of explanation much of this paper is couched in the language of queues; many of the models considered would be appropriate for physical processes not usually associated with queues.

2. Preliminary results

This section records some elementary results for linear migration processes; these processes have been discussed by Bartle [1]. The results will be used in later sections to define constants needed there.

Consider a system of $J$ colonies populated with individuals who move independently between these colonies. Suppose that an individual in colony $j$ moves to colony $k$ with probability intensity $\lambda_{jk}$.

\textbf{Remark 1.} If there is a single individual in the system then we have a Markov process with $J$ states, where state $j$ corresponds to the individual being in colony $j$. Thus we can find non-negative numbers $\alpha_1, \alpha_2, \ldots, \alpha_J$ such that
\[ \alpha_j \sum_k \lambda_{jk} = \sum_k \alpha_k \lambda_{kj} \quad j = 1, 2, \ldots, J \]

and such that \( \alpha_j > 0 \) if state \( j \) is not transient. Note that if the process has more than one irreducible class \( x_1, x_2, \ldots, x_J \) will not be unique even up to a multiplying factor.

**Remark 2.** Suppose now that in addition to migration between colonies there is immigration of an individual to colony \( j \) from outside the system with probability intensity \( \nu_j \) and an individual in colony \( j \) emigrates from the system with probability intensity \( \mu_j \). Suppose also that from any colony it is possible for an individual to emigrate from the system either directly or indirectly via a chain of other colonies; thus no colony or subset of colonies can hold an individual indefinitely. Then the equations

\[ \alpha_j \left( \mu_j + \sum_k \lambda_{jk} \right) = \nu_j + \sum_k \alpha_k \lambda_{kj} \quad j = 1, 2, \ldots, J \]

have a unique solution for \( \alpha_1, \alpha_2, \ldots, \alpha_J \). We can interpret \( \alpha_j \) as the expected number of individuals in colony \( j \) when the process is in equilibrium [11]. This interpretation shows that \( \alpha_j > 0 \) if it is possible for an individual to be in colony \( j \) when the process is in equilibrium. Note also that

\[ \sum_j \alpha_j \mu_j = \sum_j \nu_j. \]

This follows from (4) by summing over \( j \).

### 3. The equilibrium distribution for networks of queues

We now return to the process \( C \) introduced in Section 1; initially we shall consider a closed system. Let \( T_{jklm} \) be the operator which transforms a state \( C \) (where \( j, k = 1, 2, \ldots, J; \ l = 1, 2, \ldots, n_j; \ m = 1, 2, \ldots, n_k + 1 \)) by

(i) moving the customers in queue \( k \) occupying positions \( m, m + 1, \ldots, n_k \) to positions \( m + 1, m + 2, \ldots, n_k + 1 \) respectively,

(ii) moving the customer in position \( l \) in queue \( j \) to position \( m \) in queue \( k \)

(iii) moving the customers in queue \( j \) occupying positions \( l + 1, l + 2, \ldots, n_j \) to positions \( l, l + 1, \ldots, n_j - 1 \) respectively.

(Step (i) is redundant if \( m = n_k + 1 \); step (iii) is redundant if \( l = n_j \)). Suppose that a change of state of the system corresponds to the application of some operator \( T_{jklm} \) and suppose that

\[ q(C, T_{jklm}C) = \lambda_{jk}(c_j(l))\gamma_j(l, n_j)\delta_k(m, n_k + 1)\phi_j(n_j) \]

where \( \sum_k \lambda_{jk}(i) = \lambda_j \) for \( i = 1, 2, \ldots, I; \ \sum_{i=1}^{n_j} \gamma_j(l, n_j) = 1; \ \sum_{m=1}^{n_k+1} \delta_k(m, n_k+1) = 1 \) and \( \phi_j(n_j) > 0 \) if \( n_j > 0 \).

\( \phi_j(n_j) \) can be regarded as the total service effort applied in queue \( j \) while that queue contains \( n_j \) customers; \( \gamma_j(l, n_j) \) can be regarded as the proportion of this
effort directed to the customer in position \( l \) (\( l = 1, 2, \ldots, n_j \)). When a customer of type \( i \) in queue \( j \) has his service completed he moves to queue \( k \) (\( k = 1, 2, \ldots, J \)) with probability \( \lambda_{jk}(i)/\lambda_j \); on arrival at queue \( k \) he moves into position \( m \) (\( m = 1, 2, \ldots, n_k + 1 \)) within that queue with probability \( \delta_{k}(m, n_k + 1) \). Note that the type of a customer does not affect his passage through a queue, but it does affect the choice of the next queue he should join.

For example, if

\[
\begin{align*}
\phi_j(n_j) &= \min(r, n_j), \\
\gamma_j(l, n_j) &= 1/n_j \quad n_j = 1, 2, \ldots, r; \; l = 1, 2, \ldots, n_j, \\
&= 1/r \quad n_j = r + 1, r + 2, \ldots; \; l = 1, 2, \ldots, r, \\
&= 0 \quad \text{otherwise}, \\
\delta_j(l, n_j) &= 1 \quad l = n_j, \\
&= 0 \quad \text{otherwise},
\end{align*}
\]

then queue \( j \) is essentially an \( r \)-server queue in which customers have their service commenced in the order of their arrival and each customer has an exponentially distributed service time with mean \( 1/\lambda_j \).

We can use \( \lambda_{jk}(i) \) (\( j, k = 1, 2, \ldots, J \)) to define \( \alpha_1(i), \alpha_2(i), \ldots, \alpha_J(i) \) by an appeal to Remark 1 of Section 2. Note that if the process considered in that remark (now with \( \lambda_{jk} = \lambda_{jk}(i), j = 1, 2, \ldots, J \)) has more than one irreducible class then type \( i \) customers will not be able to pass between certain subsets of the \( J \) queues; hence states in \( \mathcal{P} \) will have a fixed number of type \( i \) customers contained within these subsets. Note also that if there is a state in \( \mathcal{P} \) which allows a type \( i \) customer to be in queue \( j \) then \( \alpha_j(i) > 0 \). Let

\[
A_j(c_j) = \prod_{l=1}^{n_j} \frac{\alpha_j(c_j(l))}{\phi_j(l)}
\]

if \( n_j \geq 1 \) and let \( A_j(c) = 1 \). We see that if \( C \in \mathcal{P} \) then \( A_j(c_j) > 0 \), \( j = 1, 2, \ldots, J \).

**Theorem 1.** The unique equilibrium distribution over \( \mathcal{P} \) for the closed network of queues described above is of the form

\[
P(C) = b \prod_{j=1}^{J} A_j(c_j)
\]

where \( b \) is a positive constant.

**Proof.** Equation (1) for the system under consideration can be written
Assume now that \( P(C) \) is of the form stated in the theorem. Then from expressions (7) and (6) we can deduce that

\[
P(T_{jklm}C) = P(C).
\]

Using this expression and Equation (3) we can obtain, with a modicum of algebra, the partial balance equations (cf. Whittle [10])

\[
\sum_{k} \sum_{l=1}^{n_j} \sum_{m=1}^{n_{k+1}} \lambda_{jk}(c_j(l))\gamma_{j}(l, n_j) \delta_{k}(m, n_k + 1)\phi_{j}(n_j)P(C)
\]

\[
= \sum_{j} \sum_{k} \sum_{l=1}^{n_j} \sum_{m=1}^{n_{k+1}} \lambda_{jk}(c_j(l))\gamma_{j}(m, n_j) \delta_{k}(l, n_j)\phi_{j}(n_k + 1)P(T_{jklm}C).
\]

for \( j \) such that \( n_j > 0 \). Equation (8) follows from these equations.

Now \( \prod_{j=1}^{J} A_j(c_j) > 0 \) for \( C \in \mathcal{S} \) and \( \mathcal{S} \) is finite; hence \( b \) can be chosen so that Equation (2) is satisfied by \( P(C) = b \prod_{j=1}^{J} A_j(c_j) \). This completes the proof of the theorem.

Remark. The theorem does not imply that \( c_1, c_2, \ldots, c_J \) are independent because \( C \) is restricted to lie in \( \mathcal{S} \).

We shall next consider an open system, in which customers are able to enter and leave the network of queues. Let \( T_{j, l} \) be the operator which transforms a state \( C \) (where \( j = 1, 2, \ldots, J; \ l = 1, 2, \ldots, n_j \)) by

(i) removing the customer in position \( l \) in queue \( j \)
(ii) moving the customers in queue \( j \) occupying positions \( l + 1, l + 2, \ldots, n_j \) to positions \( l, l + 1, \ldots, n_j - 1 \), respectively

and let \( T_{k, m} \) transform \( C \) (where \( k = 1, 2, \ldots, J; \ m = 1, 2, \ldots, n_k + 1 \)) by

(i) moving the customers in queue \( k \) occupying positions \( m, m + 1, \ldots, n_k \) to positions \( m + 1, m + 2, \ldots, n_k + 1 \) respectively
(ii) introducing a customer of type \( i \) at position \( m \) in queue \( k \).
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\[ q(C, T_{jklmC}) = \lambda_{jk}(c(l))\gamma_{j}(l, n_j)\delta_{k}(m, n_k + 1)\phi_{j}(n_j) \]

where

\[ \sum_{k} \lambda_{jk}(i) + \mu_{j}(i) = \lambda_{j} \text{ for } i = 1, 2, \cdots, I, \]

\[ \sum_{i=1}^{n_j} \gamma_{j}(l, n_j) = 1, \sum_{m=1}^{n_k+1} \delta_{k}(m, n_k + 1) = 1 \text{ and } \phi_{j}(n_j) > 0 \text{ if } n_j > 0. \]

The functions \( \phi_{j}(n_j), \gamma_{j}(l, n_j), \delta_{k}(m, n_k + 1) \) can be interpreted in the same manner as in the closed system. Arrivals from outside the system of type \( i \) customers at queue \( k \) form a Poisson process of rate \( v_{k}(i) \) and as \( i \) and \( k \) vary they index independent Poisson processes. When a customer of type \( i \) in queue \( j \) has his service completed there is now a probability \( \mu_{j}(i)/\lambda_{j} \) that he will leave the system.

It is possible that the transition rates (9) will not allow the system to reach an equilibrium; for example customers may enter the system at too fast a rate for them to be served. Assume for the moment that this is not the case — conditions for an equilibrium to exist will be given in Theorem 2.

Let us assume that \( \lambda_{jk}(i), j, k = 1, 2, \cdots, J, \mu_{j}(i), v_{j}(i), j = 1, 2, \cdots, J \) are such that a customer of type \( i \) in any queue can eventually leave the system either directly or indirectly via a chain of other queues (observe that this property does not depend upon the form of the functions \( \phi_{j}(n_j), \gamma_{j}(l, n_j) \) and \( \delta_{k}(m, n_k + 1) \)). We can then use Remark 2 of Section 2 to define \( \alpha_{1}(i), \alpha_{2}(i), \cdots, \alpha_{J}(i) \). Let

\[ A_{j}(e) = \prod_{i=1}^{n_j} \frac{\alpha_{i}(c(l))}{\phi_{i}(l)} \]

if \( n_j \geq 1 \) and let \( A_{j}(e) = 1 \). Although we have used the same notation as in the closed case we must remember that \( \alpha_{1}(i), \alpha_{2}(i), \cdots, \alpha_{J}(i) \) are now defined in a slightly different manner.

**Theorem 2.** Let \( a_{j} = \sum_{i} \alpha_{j}(i) \). If

\[ \sum_{n=1}^{\infty} \frac{a_{j}^{n}}{\prod_{l=1}^{n} \phi_{j}(l)} < \infty \]

for \( j = 1, 2, \cdots, J \) then the open network of queues described above has a unique equilibrium distribution and this distribution is of the form

\[ P(C) = b \prod_{j=1}^{J} A_{j}(e_{j}) \]

where \( b \) is a positive constant.
Proof. Assume that $P(C)$ is of the form stated in the theorem. Then using Equations (4) and (5) we can obtain, with a modicum of algebra, the partial balance equations (cf. Whittle [11])

$$
\sum_{j} \sum_{k} \lambda_{jk}(c_j(l))\gamma_j(l, n_j)\delta_k(m, n_k + 1)\phi_j(n_j)P(C) + \sum_{i} \mu_i(c_i(l))\gamma_i(l, n_i)\phi_i(n_i)P(C)
$$

and

$$
\sum_{i} \sum_{k} v_i(c_i(l))\delta_j(l, n_j)P(T_{jk,l}C)
$$

These equations imply

$$
\sum_{j: n_j > 0} \sum_{k} \sum_{m=1}^{n_k + 1} \lambda_{jk}(c_j(l))\gamma_j(l, n_j)\delta_k(m, n_k + 1)\phi_j(n_j)P(C) + \sum_{i} \sum_{j: n_j > 0} \mu_j(c_j(l))\gamma_j(l, n_j)\phi_j(n_j)P(C)
$$

and hence that

$$
P(C) = \sum_{D} q(C, D) \sum_{D} P(D)q(D, C)
$$

for any state $C$ where the summation extends over all states.

Now

$$
\sum_{C} \prod_{j=1}^{J} A_j(c_j) = \prod_{j=1}^{J} \sum_{c_j} A_j(c_j)
$$

(since all terms are non-negative)

$$
= \prod_{j=1}^{J} \sum_{n=0}^{\infty} \sum_{c_j: n_j = n} A_j(c_j)
$$

$$
= \prod_{j=1}^{J} \left\{ 1 + \sum_{n=1}^{\infty} \frac{a_j^n}{\prod_{l=1}^{n} \phi_j(l)} \right\} < \infty
$$
by the hypothesis of the theorem. Hence we can choose $b$ so that $P(C)$ satisfies

$$\sum_{C} P(C) = 1. \tag{12}$$

From Equations (11) and (12) it follows that $P(C)$ is an equilibrium distribution for the process, and that there exists at least one irreducible class. For $P(C)$ to be the unique equilibrium distribution we need to show that there is only one irreducible class. But from any state $C$ it is possible by a series of transitions to reach the state $(e,e,\ldots,e)$; this can be deduced from our assumption that any customer in any queue can eventually leave the system. Thus there can be only one irreducible class, which we will call $\mathcal{P}$. Moreover $C \in \mathcal{P}$ iff $P(C) > 0$. This completes the proof of the theorem; also we have shown the following:

**Corollary.** If the open network of queues described above is in equilibrium then $c_1, c_2, \ldots, c_J$ are independent.

**Remarks.** In equilibrium $P(c_j) \propto A_j(c_j)$ and hence

$$P(n \text{ customers in queue } j) \propto \frac{a_j^n}{\prod_{l=1}^{n} \phi_j(l)}.$$ 

If $\sum_{n=1}^{\infty} [a_j^n / \prod_{l=1}^{n} \phi_j(l)]$ diverges then customers arrive at queue $j$ at a faster rate than they can be served and the process will not be able to attain equilibrium.

Networks of queues with one type of customer have previously been studied by Jackson [3] and Whittle [10], [11]. Whittle first noted the phenomenon of partial balance, which has been further discussed by Kingman [7].

### 4. Extensions

The results of the last section can be extended to include processes with slightly more general transition rates; this is done in (a) and (b). In (c), (d) and (e) we indicate how similar results can be obtained for processes with quite different transition rates; essentially by imposing further restrictions in certain directions it is possible to allow greater freedom in other directions. In this section we restrict attention to open systems; the remarks also apply, mutatis mutandis, to closed systems.

(a) We can allow $\phi_j(r)$ to be zero for some values of $j$ and $r$. Suppose that

$$\phi_j(r_j) = 0$$

$$\phi_j(l) > 0 \quad l > r_j.$$
This would correspond to the servers at queue \( j \) only operating if more than \( r_j \) customers were present. The condition for an equilibrium to exist is now that

\[
\sum_{n=r_j+1}^{\infty} \frac{a_j^n}{\prod_{l=r_j+1}^{\infty} \phi_j(l)} < \infty
\]

for \( j = 1, 2, \ldots, J \). The equilibrium distribution is again of the form (10) but where we now define

\[
A_j(c_j) = \begin{cases} 
0 & n_j < r \\
\prod_{l=1}^{r_j} x_j(c_j(l)) & n_j = r \\
\prod_{l=r_j+1}^{\infty} x_j(c_j(l)) & n_j > r.
\end{cases}
\]

(b) We can allow \( y_j(l, n_j), \delta_j(l, n_j) \) to be replaced by \( y_j(c_j), \delta_j(c_j) \) so long as \( y_j \) and \( \delta_j \) are functions invariant under permutations of \( c_j \). In place of the restrictions that \( \sum_{l=1}^{r_j} y_j(l, n_j) = 1, \sum_{l=1}^{r_j} \delta_j(l, n_j) = 1 \) we must now require that

\[
\sum_{l=1}^{r_j} y_j(l, c_j) = 1, \quad \sum_{l=1}^{r_j} \delta_j(l, c_j) = 1.
\]

(c) Suppose that the open system described in the last section is amended by restricting the functions \( y_j, \delta_j \) to be identical (i.e., \( y_j(l, n_j) = \delta_j(l, n_j), n_j = 1, 2, \ldots; l = 1, 2, \ldots, n_j \)) for \( j = 1, 2, \ldots, J \), and by allowing \( \sum_{k} \lambda_{jk}(i) + \mu_{j}(i) \) to vary with \( i \), say \( \sum_{k} \lambda_{jk}(i) + \mu_{j}(i) = \lambda_{j}(i), i = 1, 2, \ldots, l; j = 1, 2, \ldots, J. \) Then we find that the equilibrium distribution, if it exists, is of the form (10). The passage of a customer through a queue is now affected by his type; \( \lambda_{j}(i) \) can be regarded as a measure of the expected amount of service required by a customer of type \( i \) at queue \( j \) and this can now vary with the type of the customer.

As an example suppose that \( y_j(l, n_j) = \delta_j(l, n_j) = 1/n_j \) for \( l = 1, 2, \ldots, n_j; j = 1, 2, \ldots, J \). Thus the service effort within a queue is equally spread over all the customers present, and the order of the customers in a queue is independent of the order of arrival of these customers; in fact if we know which customers are in each queue then the order of these customers within each queue will tell us nothing more about the past or future behaviour of the system. The probability intensity that a customer of type \( i \) in queue \( j \) will leave that queue is \( \lambda_{j}(i)\phi_j(n_j)/n_j \) irrespective of the customer’s position in the queue. We can consider this example to be a generalization of the linear migration processes discussed in Section 2 and of the non-linear migration processes discussed by Whittle [10], [11]. An ‘individual’ now has a type which influences his path through the ‘colonies’ and his length of stay in any particular ‘colony’.

(d) Suppose that the open system described in the last section is amended by
(i) requiring that the functions $\gamma_j, \delta_j$ be identical to the function $\zeta$ for $j = 1, 2, \cdots, J$ where

\begin{equation}
\zeta(l, n) = \begin{cases} 
1 & \text{if } l = n \\
0 & \text{otherwise}
\end{cases}
\end{equation}

(ii) allowing $\sum_k \lambda_{jk}(i) + \mu_j(i)$ to vary with $i$ for $j = 1, 2, \cdots, J$

(iii) replacing the function $\phi_j(n_j)$ by the function $\phi_j(c_j)$ for $j = 1, 2, \cdots, J$.

Thus customers arrive at, and leave from, the last position in a queue. Each such queue could be considered as a stack, with ‘items’ or ‘particles’ arriving at, and departing from, the ‘top’ of the stack. The probability intensity that the item at the top of stack $j$ will move to the top of stack $k$ is $\lambda_{jk}(c_j(n_j))\phi_j(c_j)$; the function $\phi_j(c_j)$ allows this intensity to depend upon $c_j$ and not just upon $n_j$.

If we define

\begin{equation}
A_j(c_j) = \prod_{i=1}^{n_j} \frac{\alpha_j(c_j(i))}{\phi_j(c_j(1), c_j(2), \cdots, c_j(l))}
\end{equation}

then we find that form (10) gives the equilibrium distribution, if it exists.

(e) The process is reversible if (Reich [9])

\[ P(C)q(C, D) = P(D)q(D, C) \quad C, D \in \mathcal{H}. \]

If we impose conditions which ensure that the process $C$ is reversible then we can extend the results of the last section in many directions. For example suppose that

\begin{align*}
\alpha_j(i)\lambda_{jk}(i) &= \alpha_k(i)\lambda_{kj}(i) \\
\alpha_j(i)\mu_j(i) &= \nu_j(i)
\end{align*}

for $i = 1, 2, \cdots, n_j$; $j, k = 1, 2, \cdots, J$. Consider the transition rates

\begin{align*}
q(C, T_{j,i}) &= \mu_j(c_j(l))\gamma_j(l, n_j)\phi_j(n_j) \\
q(C, T_{k,m}) &= \nu_k(i)\gamma_k(m, n_k + 1)\psi_k(n_k + 1) \\
q(C, T_{jklm}) &= \lambda_{jk}(c_j(l))\gamma_j(l, n_j)\gamma_k(m, n_k + 1)\phi_j(n_j)\psi_k(n_k + 1).
\end{align*}

If we define

\begin{equation}
A_j(c_j) = \prod_{i=1}^{n_j} \frac{\alpha_j(c_j(i))\psi_j(i)}{\phi_j(i)}
\end{equation}

then we find that form (10) gives the equilibrium distribution, if it exists, and that the process is reversible. This process is a generalization of one discussed by Kingman [7].
5. The effect of time reversal

If \( C(t) \) is a Markov process in equilibrium then the reversed process \( C(-t) \) is also a Markov process in equilibrium ([8] p. 138, [4] p. 136). The reversed process has the same equilibrium distribution \( P(C) \) over \( \mathcal{S} \), but its transition rates will in general be different from those of the original process. If \( q'(C,D) \) denotes the transition rate from \( C \) to \( D \) of the reversed process then

\[
P(C)q'(C, D) = F(D)q(D, C) \quad C, D \in \mathcal{S}
\]

(Note that \( P(C) > 0 \) since \( C \in \mathcal{S} \). Also \( q'(C, D) = q(C, D) \) for \( C, D \in \mathcal{S} \) iff the process in equilibrium is reversible.) In this section we shall investigate in detail the effect of time reversal on the open process considered in Section 3. Related results can be obtained for the closed process of that section and for the processes discussed in Section 4.

**Theorem 3.** The reversed process obtained from the open network of queue of Section 3 is also an open network of queues.

**Proof.** Using Equation (14) we can determine the transition rates of the reversed process.

\[
q'(C, T_{jklm}^C) = q(T_{jklm}^C, C) P(T_{jklm}^C)/P(C)
\]

\[
= \lambda_j^C(c_j(l)) \gamma_k(m, n_k + 1) \delta_j(l, n_j) \phi_k(n_k + 1) \gamma_k(c_j(l)) \phi_k(n_k + 1)
\]

\[
= \mu_j^C(c_j(l)) \gamma_k(m, n_k + 1) \delta_j(l, n_j) \phi_k(n_j) \gamma_k(c_j(l)) \phi_k(n_k + 1)
\]

\[
q'(C, T_{j,l}^C) = q(T_{j,l}^C, C) P(T_{j,l}^C)/P(C)
\]

\[
= \nu_j(c_j(l)) \delta_j(l, n_j) \phi_j(n_j) \gamma_j(c_j(l))
\]

\[
q'(C, T_{k,m}^C) = q(T_{k,m}^C, C) P(T_{k,m}^C)/P(C)
\]

\[
= \mu_k(c_k(l)) \gamma_k(m, n_k + 1) \phi_k(n_k + 1) \gamma_k(c_k(l)) \phi_k(n_k + 1)
\]

Define

\[
\lambda_j^C(i) = \gamma_k(i) \lambda_k^C(i) / \gamma_j(i) \quad \text{if} \quad \gamma_j(i) > 0
\]

\[
= 0 \quad \text{if} \quad \gamma_j(i) = 0
\]

\[
\mu_j^C(i) = \gamma_k(i) \mu_k(i) \quad \text{if} \quad \gamma_j(i) > 0
\]

\[
= 0 \quad \text{if} \quad \gamma_j(i) = 0
\]

\[
\nu_j(i) = \gamma_k(i) \mu_k(i)
\]

and define \( \gamma_j^C \equiv \delta_j, \delta_k^C \equiv \gamma_k \) for \( i = 1, 2, \ldots, I; j, k = 1, 2, \ldots, J \). Then
Networks of queues with customers of different types

\[ q'(C, T_{l,n}) = \mu_j(c_j(l))\phi_j(n_j) \]

\[ q'(C, T_{l,m}) = \nu_k(c_k(l))\phi_k(n_k) \]

\[ q'(C, T_{l,m,n}) = \lambda_j(c_j(l))\phi_j(n_j) \]

We can show that \( \lambda_j(i), \mu_j(i), \nu_k(i), \gamma_j(i) \) and \( \delta_k(i) \) for \( i = 1, 2, \ldots, I; j, k = 1, 2, \ldots, J \) satisfy all the conditions imposed in Section 3 and hence, comparing (16) with (9), we see that the reversed process is also an open process of the form considered in Section 3.

**Remarks.** We can use Theorem 3 to prove indirectly results about the original process. For example, note that arrivals from outside the system for the reversed process correspond to departures from the system for the original process. Hence in the original process departures from the system of type \( i \) customers from queue \( k \) form a Poisson process of rate \( \nu_k(i) = \alpha_k(i)\mu_k(i) \) and as \( i \) and \( k \) vary they index independent Poisson processes. Also it follows that the present state of the process is independent of past departures from the system.

As another illustration of the use of a reversed process to obtain information about an original process consider the example described in (c) of Section 4. As noted there the order of a queue is independent of the order of arrival of the customers. Suppose now that we are interested in the order of arrival. We can show that the reversed process is of the same form as the original process but with \( \lambda_j(i), \mu_j(i), \nu_k(i) \) replaced by \( \lambda'_j(i), \mu'_j(i), \nu'_k(i) \) as defined in (15), for \( i = 1, 2, \ldots, I; j, k = 1, 2, \ldots, J \). Note that \( \lambda'_j(i) = \sum_k \lambda'_j(i) + \mu'_j(i) = \lambda_j(i) \), at least when \( \alpha_k(i) > 0 \). Now the order of departure from queue \( j \) in the reversed process corresponds to the reverse of the order of arrival in the original process and thus we can show that if in the original process queue \( j \) contains customers of type \( i_1, i_2, \ldots, i_n \) then the probability that \( (i_1, i_2, \ldots, i_n) \) is the order of their arrival is

\[ \frac{\lambda_j(i_1) \cdots \lambda_j(i_{n-1})}{\sum_{r=1}^{n} \lambda_j(i_r)} \cdot \frac{\lambda_j(i_n)}{\sum_{r=1}^{n} \lambda_j(i_r)} = \frac{\lambda_j(i_1) \cdots \lambda_j(i_n)}{\sum_{r=1}^{n} \lambda_j(i_r)} . \]

We can verify this result in another way. Replace \( \delta_j \) in the original process by \( \xi \) (as defined in (13)) for \( j = 1, 2, \ldots, J \); hence we will have a process in which the order of a queue is the order of arrival. We can check, using Equations (1) and (2), that the equilibrium distribution for this amended process is

\[ P(C) \propto \prod_{j=1}^{J} \left\{ A_j(c_j) \prod_{l=1}^{n_j} \frac{\lambda_j(c_j(l))}{\sum_{r=1}^{n_j} \lambda_j(c_j(r))} \right\} . \]

Reich [9] observed that the reversibility of a process enables one to make powerful statements about the behaviour in equilibrium of that process. In this
section we have shown that if the reversed process is of a familiar form then this tells us much about the process.

Acknowledgements

I should like to thank Professor P. Whittle for his advice and encouragement, and the referee for some helpful comments on an earlier draft of this paper. I am grateful to the Science Research Council for its financial support.

References