Case-study: Modeling and Performance Analysis of BitTorrent Peer-to-Peer Networks

690PE Fall 2016
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In previous lectures

Lecture 18: Fluid modeling: mean-field

Lecture 19: Fluid modeling: Poisson-driven SDE, on-off fluid source

Lecture 20: Fluid modeling: superposition of on-off fluid sources, ad hoc models

Lectures 21-22: Modeling of TCP
In today's lecture


Learning objectives:

• understand key mechanisms in BitTorrent peer-to-peer systems

• use differential equations (fluid model) as an approximate model for complex systems

• determine stability of systems based on differential equations that describe it
Addendum

Tracker sends list containing only a subset of nodes to each new peer.
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New peer establishes connections to all peers in the list
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Some connections will be used for data transfer too.
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Some connections will be used for data transfer too

If a peer “unchokes” me, I will ask him the rarest piece first among my connections
Assume equilibrium exists:

\[
0 = \lambda - \theta \bar{x} - \min\{c \bar{x}, \mu(\eta \bar{x} + \bar{y})\}, \\
0 = \min\{c \bar{x}, \mu(\eta \bar{x} + \bar{y})\} - \gamma y(t),
\]  

(2)
Assume equilibrium exists:

\[ 0 = \lambda - \theta \bar{z} - \min\{c \bar{x}, \mu(\eta \bar{x} + \bar{y})\}, \]
\[ 0 = \min\{c \bar{x}, \mu(\eta \bar{x} + \bar{y})\} - \gamma y(t), \]  \hspace{1cm} (2)

Assume \( \eta > 0 \)
Assume equilibrium exists:

\[ 0 = \lambda - \theta \bar{x} - \min\{c\bar{x}, \mu(\eta \bar{x} + \bar{y})\}, \]
\[ 0 = \min\{c\bar{x}, \mu(\eta \bar{x} + \bar{y})\} - \gamma y(t), \]  

(2)

Assume \( \eta > 0 \)

Case 1: \( c\bar{x} \leq \mu(\eta \bar{x} + \bar{y}) \)

\[ \bar{x} = \frac{\lambda}{c(1 + \frac{a}{c})} \]
\[ \bar{y} = \frac{\lambda}{\gamma(1 + \frac{a}{c})}. \]  

(3)
Steady-State equilibrium

Assume equilibrium exists:

\[ 0 = \lambda - \theta \bar{x} - \min\{c\bar{x}, \mu(\eta \bar{x} + \bar{y})\}, \]
\[ 0 = \min\{c\bar{x}, \mu(\eta \bar{x} + \bar{y})\} - \gamma y(t), \quad (2) \]

Assume \( \eta > 0 \)

Case 1: \( c\bar{x} \leq \mu(\eta \bar{x} + \bar{y}) \) \( \Rightarrow \frac{1}{c} \geq \frac{1}{\eta} \left( \frac{1}{\mu} - \frac{1}{\gamma} \right) \]

\[ \bar{x} = \frac{\lambda}{c(1 + \frac{\eta}{c})} \]
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Assume \( \eta > 0 \)

Case 1: \( c\bar{x} \leq \mu(\eta \bar{x} + \bar{y}) \) \( \Rightarrow \frac{1}{c} \geq \frac{1}{\eta} \left( \frac{1}{\mu} - \frac{1}{\gamma} \right) \)

\[ \bar{x} = \frac{\lambda}{c(1 + \frac{\mu}{c})} \]
\[ \bar{y} = \frac{\lambda}{\gamma(1 + \frac{\mu}{c})}. \]  \hspace{1cm} (3)

Case 2: \( c\bar{x} > \mu(\eta \bar{x} + \bar{y}) \) \( \Rightarrow \frac{1}{c} < \frac{1}{\eta} \left( \frac{1}{\mu} - \frac{1}{\gamma} \right) \)
Average download time (those who don’t abandon)?

\[
\theta x(t) \\
\min\{cx(t), \mu(\eta x(t) + y(t))\}
\]
Little’s Law

Average download time (those who don’t abandon)?

$$\theta \bar{x}(t)$$

Effective arrival rate:

$$\lambda - \theta \bar{x}$$

Fraction who don’t abandon:

$$\frac{\lambda - \theta \bar{x}}{\lambda}$$
Little’s Law

Average download time (those who don’t abandon)?

\[ \lambda \]

\[ \min \{ cx(t), \mu(\eta x(t) + y(t)) \} \]

\[ \theta x(t) \]

Effective arrival rate:

\[ \lambda - \theta \bar{x} \]

Fraction who don’t abandon:

\[ \frac{\lambda - \theta \bar{x}}{\lambda} \]

\[ \frac{\lambda - \theta \bar{x}}{\lambda} \bar{x} = (\lambda - \theta \bar{x})T \]

\[ \frac{\bar{x}}{\bar{x}} = T \]
Each leecher is connected to $k = \min\{x - 1, K\}$ peers.

Peer $i$ has pieces of interest to $1+$ peers connected to it. $\Rightarrow$ peer $i$ will upload data.
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Peer $i$ has pieces of interest to $1+$ peers connected to $i$. \Rightarrow peer $i$ will upload data

$$\eta = 1 - \mathbb{P}\left(\text{downloader } i \text{ has no piece that the connected peers need}\right).$$
Each leecher is connected to \( k = \min\{x - 1, K\} \) peers.

Peer \( i \) has pieces of interest to \( 1+ \) peers connected to it \( \Rightarrow \) peer \( i \) will upload data.

\[
\eta = 1 - \Pr\left\{ \text{downloader } i \text{ has no piece that the connected peers need} \right\}.
\]

Assumption: piece distributions between peers are i.i.d.
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Peer \( i \) has pieces of interest to \( 1+ \) peers connected to it \( \rightarrow \) peer \( i \) will upload data

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Assumption: piece distributions between peers are i.i.d.

\[
\eta = 1 - \mathbb{P}\left\{ \text{downloader } j \text{ needs no piece from downloader } i \right\}^k.
\]

\[
downloader j \text{ needs no piece from downloader } i \right\} = \mathbb{P}\{j \text{ has all pieces of download} \}.
\]
Let $n_i$ be the number of pieces peer $i$ has.

Assumption: $n_i \sim \text{uniform in } \{0, \ldots, N - 1\}$.

Assumption (rarest-first): Given $n_i$, these pieces are randomly chosen.
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\{ $j$ has all pieces of downloader $i$ \}
Effectiveness of File Sharing (2/4)

Let $n_i$ be the number of pieces peer $i$ has.

Assumption: $n_i \sim \text{uniform in } \{0, \ldots, N - 1\}$.

Assumption (rarest-first): Given $n_i$, these pieces are randomly chosen.

\[ \{j \text{ has all pieces of downloader } i\} \]

\[
= \sum_{n_j=1}^{N-1} \sum_{n_i=0}^{n_j} \frac{1}{N^2} \mathbb{P}\{j \text{ has all pieces of } i | n_i, n_j\}
\]

\[
= \sum_{n_j=1}^{N-1} \sum_{n_i=0}^{n_j} \frac{1}{N^2} \frac{(N-n_i)}{\binom{N}{n_j}}
\]
Let $n_i$ be the number of pieces peer $i$ has.

Assumption: $n_i \sim$ uniform in $\{0, \ldots, N - 1\}$.

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\]

\[
= \sum_{n_j=1}^{N-1} \sum_{n_i=0}^{n_j} \frac{1}{N^2} \frac{\binom{N-n_i}{n_j}}{\binom{N}{n_j}}
\]

\[
\sum_{n_i=0}^{n_j} \binom{N-n_i}{n_j} = \binom{N+1}{n_j}
\]

Chu-Vandermonde identity.
$j$ has all pieces of downloader $i$} \\
= \sum_{n_j=1}^{N-1} \frac{1}{N^2} \frac{\binom{N+1}{n_j}}{\binom{N}{n_j}} \\
= \sum_{n_j=1}^{N-1} \frac{N+1}{N^2(N+1-n_j)} \\
= \frac{N+1}{N^2} \sum_{n_j=1}^{N-1} \frac{1}{N+1-n_j} \\
= \frac{N+1}{N^2} \sum_{m=2}^{N} \frac{1}{m} \approx \frac{\log N}{N}$
\[ \eta = 1 - P \left\{ \text{downloader } j \text{ needs no piece from downloader } i \right\}^k, \]

\[ \approx 1 - \left( \frac{\log N}{N} \right)^k. \]  \hspace{1cm} (7)

Suppose file size 1GB, piece size 256KB \( \Rightarrow N = 4,000 \)

Then for \( k = 1, \eta \approx 0.998. \)
Physical stability of system of differential equations:

First-order equations:

\[ \dot{x} = f(t, x), \quad t \geq 0, \quad x(0) = x_0 \]

A solution \( \phi(t) \) is stable if every solution \( \psi(t) \) close to \( \phi(t) \) at \( t = 0 \) remains close for all \( t > 0 \).
Physical stability of system of differential equations:

\[ \dot{x} = f(t, x), \; t \geq 0, \; x(0) = x_0 \]

A solution \( \phi(t) \) is stable if every solution \( \psi(t) \) close to \( \phi(t) \) at \( t = 0 \) remains close for all \( t > 0 \).

Formally, for each \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( |\psi(t) - \phi(t)| < \epsilon \) whenever \( |\psi(0) - \phi(0)| < \delta \).
Physical stability of system of differential equations:

\[ \dot{x} = f(t, x), \quad t \geq 0, \quad x(0) = x_0 \]

Solution \( \phi(t) \) is stable if every solution \( \psi(t) \) close to \( \phi(t) \) at \( t = 0 \) remains close for all \( t > 0 \).

Formally, for each \( \epsilon > 0 \) there is a \( \delta > 0 \) such that

\[ |\psi(t) - \phi(t)| < \epsilon \]

whenever \( |\psi(0) - \phi(0)| < \delta \).

In general, depends greatly on \( f(t, x) \), may be intractable. Expression doesn’t depend explicitly on \( t \), analysis is tractable.
Stability (cont'd)

\[
Df(x) = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}
\]

- matrix of 1st order partial derivatives of \( f((x)) \) evaluated at \( c \)
- eigenvalues of \( Df(c) \):
  - all have negative real part \( \Rightarrow \) every sol. stable
  - at least one has positive real part \( \Rightarrow \) every sol. unstable
Let $A$: $n \times n$ matrix.

$$Au = \omega u$$

$u \in \mathbb{R}^n$ is an eigenvector of $A$

associated with eigenvalue $\omega \in \mathbb{C}$. 
Eigenvalues and characteristic equation

Let $A$: $n \times n$ matrix.

$$Av = \omega v$$

$v \in \mathbb{R}^n$ is an eigenvector of $A$

Associated with eigenvalue $\omega \in \mathbb{C}$.

$$(A - \omega I)v = 0$$
Eigenvalues and characteristic equation

Let $A$: $n \times n$ matrix.

$$Av = \omega v$$

$v \in \mathbb{R}^n$ is an eigenvector of $A$ associated with eigenvalue $\omega \in \mathbb{C}$.

$$(A - \omega I)v = 0$$

Vector $0$ is not considered an eigenvector. Hence, eigenvalues correspond to the solutions of

$$\det(A - \omega I) = 0,$$

which gives rise to the characteristic equation.
\[ \frac{1}{c} < \frac{1}{\eta} \left( \frac{1}{\mu} - \frac{1}{\gamma} \right): \]

\[ \psi^2 + (\mu \eta + \theta + \gamma - \mu)\psi + \mu\eta\gamma + \theta(\gamma - \mu) = 0 \]

\[ \frac{1}{c} < \frac{1}{\eta} \left( \frac{1}{\mu} - \frac{1}{\gamma} \right), \text{ we have } \gamma > \mu. \]

\[ \text{If } \eta > 0, \mu \eta + \theta + \gamma - \mu > 0 \text{ and } \mu \eta \gamma + \theta(\gamma - \mu) > 0, \text{ eigenvalues have strictly negative real parts } \Rightarrow \text{ stable.} \]
Stability analysis

\[ \frac{1}{c} < \frac{1}{\eta} \left( \frac{1}{\mu} - \frac{1}{\gamma} \right): \]
\[ \psi^2 + (\mu \eta + \theta + \gamma - \mu) \psi + \mu \eta \gamma + \theta (\gamma - \mu) = 0 \]
\[ \frac{1}{c} < \frac{1}{\eta} \left( \frac{1}{\mu} - \frac{1}{\gamma} \right), \text{ we have } \gamma > \mu. \]

If \( \eta > 0, \mu \eta + \theta + \gamma - \mu > 0 \) and \( \mu \eta \gamma + \theta (\gamma - \mu) > 0 \),

the eigenvalues have strictly negative real parts \( \Rightarrow \) stable.

\[ \frac{1}{c} > \frac{1}{\eta} \left( \frac{1}{\mu} - \frac{1}{\gamma} \right): \]
\[ \psi^2 + (\theta + \gamma + c) \psi + (\theta + c) \gamma = 0 \]

the eigenvalues have strictly negative real parts \( \Rightarrow \) stable.