Markovian queueing networks

690PE_16 Class 11
Consider a queueing network composed of K stations (or nodes)

- Each station is equipped with a single server and an infinite waiting room
- A customer leaving station i instantaneously joins station j=1,...,K with routing probability \( p(i,j) \) or leaves the network with routing probability \( p(i,0) \).

\[ P := [p_{i,j}]_{0 \leq i, j \leq K} \text{ with } \sum_{0 \leq j \leq K} p(i,j) = 1 \text{ for all } i = 1,...,K \]

- Customers join station \( i = 1,...,K \) from outside the network according to a Poisson process with rate \( \lambda_i^0 \geq 0 \)

- At station \( i = 1,...,K \) service times are exponentially distributed with rate \( \mu_i > 0 \)
- All arrival Poisson processes are mutually independent
- All service times are mutually independent
- All arrival Poisson processes are independent of the service times processes

Network **open** if \( \lambda_i^0 > 0 \) for at least one \( i=1,\ldots,K \)

Network **closed** if \( \lambda_i^0 = 0 \) for all \( i=1,\ldots,K \)

Such network is called a Jackson (queueing) network
Open/closed networks

open QN

closed QN
Open network

Model of CPU
Closed network

Model of CPU with user driven workload
Stationary analysis of open Jackson networks

\[ X_i(t) = 0, 1, \ldots \text{ state of node } i = 1, \ldots, K \text{ at time } t \]
\[ X(t) = (X_1(t), \ldots, X_K(t)) \text{ state of network at time } t \]

**Claim:** \( \{X(t), \, t \geq 0\} \) is a CTMC with state-space \( S = \{0, 1, \ldots\}^K \)

**Proof.** Use rule of construction for CTMC
Stationary analysis of open Jackson networks (con’t)

$e_i = \text{unit vector of dimension } K \text{ with all components equal to zero but the ith one which is one}$

Non-zero transitions from state $n = (n_1, \ldots, n_K)$:

1. $n \rightarrow n - e_i$ customer leaves network at station $i$ (provided $n_i > 0$)

2. $n \rightarrow n + e_i$ external arrival to station $i$

3. $n \rightarrow n - e_i + e_j$ customer leaving station $i$ enters station $j \neq i$ (provided $n_i > 0$)
Stationary analysis of open Jackson networks (con't)

1. \( n \rightarrow n - e_i \)
2. \( n \rightarrow n + e_i \)
3. \( n \rightarrow n - e_i + e_j \) for \( i \neq j \)

Thanks to Poisson arrivals and exponential services times, all other transitions occurring in \([t, t+h]\) are \( o(h) \) as resulting either from at least two external arrivals or from one external arrival and one service completion or from of at least two service completions
Stationary analysis of open Jackson networks (con’t)

1. \( n \rightarrow n - e_i \)

\( K(K+1) \) exponential independent rvs with rates \( \lambda_i^0, \mu_i p(i,j)1(n_i>0) \) for \( i=1,\ldots,K, j=0,\ldots,K, i \neq j \)

[Make use of thinning of a Poisson process]

Therefore, \( q(n,n-e_i) = \mu_i p(i,0)1(n_i>0) \)
Stationary analysis of open Jackson networks (con’t)

2. \( n \to n + e_i \)

\( K(K+1) \) exponential independent rvs with rates

\( \lambda_{i,0}, \mu_i p(i,j) 1(n_i > 0) \) for \( i=1,...,K, \ j=0,...,K, i\neq j \)

Therefore, \( q(n, n+e_i) = \lambda_{i,0} \)

3. \( n \to n - e_i + e_j \) for \( i\neq j \)

\( K(K+2) \) exponential independent rvs with rates

\( \lambda_{i,0}, \mu_i p(i,j) 1(n_i > 0) \) for \( i=1,...,K, \ j=0,...,K, i\neq j \)

Here, \( q(n, n-e_i+e_j) = \mu_i p(i,j) 1(n_i > 0) \) for \( i\neq j \)
Stationary analysis of open Jackson networks (con’t)

By rule of construction \{X(t), t \geq 0\} is a CTMC

Let us find see if/when one can find a unique strictly positive solution to \(xQ=0, \ x.1=1, \ x=(x(n), \ n \in S)\)

If yes, then \(x\) is steady-state distribution, i.e.

\[
\pi(n) := \lim_{t \to \infty} P(X(t)=n) = x(n) \text{ regardless of initial state}
\]
\[ \pi Q = 0 \quad \text{for all state } \underline{n} \in S \]

prob. flow leaving state \( \underline{n} = \) prob. flow entering \( \underline{n} \)

\[
\pi(\underline{n}) \left( \sum_{i=1}^{K} \lambda_i^0 + \sum_{i=1}^{K} u_i 1(n_i > 0)(1 - p(i,i)) \right) = \sum_{i=1}^{K} \lambda_i^0 1(n_i > 0) \pi(\underline{n} - e_i)
\]

\[
+ \sum_{i=1}^{K} u_i p(i,0) \pi(\underline{n} + e_i) + \sum_{j=1}^{K} \sum_{i=1 \atop i \neq j}^{K} u_i p(i, j) 1(n_j > 0) \pi(\underline{n} + e_i - e_j)
\]

Balance equations for open Jackson networks
Try a solution of the form

$$\pi(n) = C \prod_{i=1}^{K} y_i^{n_i}$$

Balance equations:

$$\pi(n + e_i) = \pi(n)y_i$$
$$\pi(n - e_i) = \pi(n) / y_i$$
$$\pi(n + e_i - e_j) = \pi(n)y_i / y_j$$

$$\pi(n) \left( \sum_{i=1}^{K} \lambda_i^0 + \sum_{i=1}^{K} \mu_i(1 - p(i,i))1(n_i > 0) \right) = \sum_{i=1}^{K} \lambda_i^0 1(n_i > 0) \pi(n-e_i)$$

$$+ \sum_{i=1}^{K} \mu_i p(i,0) \pi(n+e_i) + \sum_{j=1}^{K} \sum_{i=1}^{K} \mu_i p(i, j) 1(n_i > 0) \pi(n+e_i - e_j)$$
Try a solution of the form

$$\pi(n) = C \prod_{i=1}^{K} y_i^{n_i}$$

With

$$\pi(n + e_i) = \pi(n)y_i$$
$$\pi(n - e_i) = \pi(n) / y_i$$
$$\pi(n + e_i - e_j) = \pi(n)y_i / y_j$$

$$\pi(n) \left( \sum_{i=1}^{K} \lambda_i^0 + \sum_{i=1}^{K} \mu_i(1 - p(i,i))1(n_i > 0) \right) = \sum_{i=1}^{K} \lambda_i^0 1(n_i > 0) \pi(n) / y_i$$

$$+ \sum_{i=1}^{K} \mu_i p(i,0) \pi(n)y_i + \sum_{j=1}^{K} \sum_{i=1 \atop i \neq j}^{K} \mu_i p(i, j) 1(n_j > 0) \pi(n)y_i / y_j$$
Try a solution of the form

\[ \pi(n) = C \prod_{i=1}^{K} y_i^{n_i} \]

With

\[ \pi(n + e_i) = \pi(n) y_i \]
\[ \pi(n - e_i) = \pi(n) / y_i \]
\[ \pi(n + e_i - e_j) = \pi(n) y_i / y_j \]

\[ \pi(n) \left( \sum_{i=1}^{K} \lambda_i^0 + \sum_{i=1}^{K} \mu_i (1 - p(i,i)) 1(n_i > 0) \right) = \sum_{i=1}^{K} \lambda_i^0 1(n_i > 0) \pi(n) / y_i \]

\[ + \sum_{i=1}^{K} \mu_i p(i,0) \pi(n) y_i + \sum_{j=1}^{K} \sum_{i=1}^{K} \mu_i p(i, j) 1(n_j > 0) \pi(n) y_i / y_j \]
Left with

\[
\sum_{i=1}^{K} \lambda_i^0 + \sum_{i=1}^{K} \mu_i 1(n_i > 0)(1 - p(i,i)) = \sum_{i=1}^{K} \lambda_i^0 1(n_i > 0) / y_i \\
+ \sum_{i=1}^{K} \mu_i p(i,0) y_i + \sum_{j=1}^{K} \sum_{i=1, i\neq j}^{K} \mu_i 1(n_j > 0) p(i,j) y_i / y_j
\]

(1)

If \( y_i = \lambda_i / \mu_i \) for \( i = 1, \ldots, K \) with \( \lambda_1, \ldots, \lambda_K \) solutions of

\[
\lambda_i = \lambda_i^0 + \sum_{j=1}^{K} \lambda_j p(j,i) \quad \text{for } i = 1, \ldots, K
\]

then (1) holds (see proof in Lecture Notes, pp. 36-37)
Recap: Balance equations satisfied for

\[ \pi(n) = C \prod_{i=1}^{K} \rho_i^n \]

with \( \rho_i := \frac{\lambda_i}{\mu_i} \)

where

\[ \lambda_i = \lambda_i^0 + \sum_{j=1}^{K} \lambda_j p(j, i) \quad \text{for } i=1, \ldots, K \]

If \( \rho_i < 1 \) for all \( i=1, \ldots, K \) then \( C := (1-\rho_1) \times \cdots \times (1-\rho_K) \)

and

\[ \pi(n) = \prod_{i=1}^{K} (1 - \rho_i) \rho_i^n > 0 \quad \text{for all } n \in \{0,1,2,\ldots\}^K \]
Result (Limiting distribution open Jackson network)
If
- matrix $\mathbf{I-P}$ is invertible (irreducibility condition)
- $\rho_i := \lambda_i / \mu_i < 1$ for all $i=1, ..., K$ (stability condition)
with $\lambda_1, ..., \lambda_K$ unique positive solution of traffic eqns

$$\lambda_i = \lambda_i^0 + \sum_{j=1}^{K} \lambda_j p(j,i) \quad \text{for } i=1, ..., K$$

then

$$\pi(n) = \prod_{i=1}^{K} (1 - \rho_i) \rho_i^{n_i}, \quad n \in \{0,1,2,...\}^K,$$

is the limiting distribution of the joint number of customers in the network (i.e. $\lim_{t \to \infty} P(X(t)=n) = \pi(n)$ for all $n$) regardless of initial state.
Comments

\[ \lambda_i = \lambda_i^0 + \sum_{j=1}^{K} \lambda_j p(j, i) \quad \text{for } i=1,\ldots,K \]

\( \lambda_i \) = traffic rate entering node i in steady-state

= sum of the **exogeneous** arrival rate to node i and of traffic rates **coming from other nodes** (including i itself)
Comments

2) \( \pi(n) = \prod_{i=1}^{K} (1 - \rho_i) \rho_i^{n_i} = P(X_1=n_1, ..., X_K=n_K) \)

with \( X_1, ..., X_K \) stat. # of cust. in nodes 1, ..., K exhibits a product-form

Its shows that \( X_1, ..., X_K \) independent rvs as \( P(X_1=n_1, ..., X_K=n_K) = P(X_1=n_1) \times ... \times P(X_K=n_K) \)
with \( P(X_i=n) = (1-\rho_i)\rho_i^n \) for \( n=0,1,.... \)

\( \Rightarrow \) Stationary distribution of M/M/1 queue with arrival rate \( \lambda_i \) and service rate \( \mu_i \)
Comments

However, internal flows in Jackson open network are in general **not** Poisson processes as loops destroy the Poisson nature.

Poisson internal flows only for so-called **feedforward networks** (e.g. queues in series) as output of stationnary $M/M/1$ queue is Poisson process with same rate as that of input process.

⇒ J.R. Jackson’s result (1957) really non-intuitive!
Performance measures

\[ E[X_i] = \sum_{n_1 \geq 0} \sum_{n_2 \geq 0} ... \sum_{n_K \geq 0} n_i P(X_1 = n_1, ..., X_N = n_K) \]

\[ = \sum_{n_1 \geq 0} \sum_{n_2 \geq 0} ... \sum_{n_K \geq 0} n_i \prod_{j=1}^{K} (1 - \rho_j) \rho_j^{n_j} \]

\[ = \prod_{j=1}^{K} \sum_{n_{j \neq i} \geq 0} (1 - \rho_j) \rho_j^{n_{j \neq i}} \cdot \sum_{n_i \geq 0} n_i (1 - \rho_i) \rho_i^{n_i} \]

\[ = \frac{\rho_i}{1 - \rho_i} \]

\[ \text{Could have found it directly as node } i \text{ is behaving as } M/M/1 \text{ queue with traffic intensity } \rho_i \text{ (= it has the same prob. distribution for number of customers)} \]
Performance measures (cont')

\[ T = \text{expected sojourn time in open Jackson network of an arbitrary customer} \]

By Little's result

\[ T = \frac{\left( E[X_1] + \ldots + E[X_K] \right)}{\left( \lambda_1^0 + \ldots + \lambda_K^0 \right)} \]

\[ = \frac{1}{\sum_{i=1}^{K} \lambda_i^0} \sum_{i=1}^{K} \frac{\rho_i}{1 - \rho_i} \]

provided \( \rho_i < 1 \) for \( i=1,\ldots,K \)
Ex. 1: Smallest open Jackson network

Traffic equation: \( \lambda = \lambda^0 + p \lambda \)

- If \( p = 1 \) no solution (not irreducible, customers cannot leave the network)
- If \( p < 1 \) \( \lambda = \lambda^0/(1-p) \)

Limiting distribution \( \pi(n) = (1-\rho)\rho^n \), \( n \geq 0 \), provided \( \rho := \lambda^0/(\mu(1-p)) < 1 \) (stability) and \( p < 1 \) (irreducibility)
Ex. 1: Smallest open Jackson network

Limiting distribution $\pi(n) = (1-\rho)\rho^n$, $n \geq 0$, provided $\rho := \frac{\lambda^0}{\mu(1-p)} < 1$ (stability) and $p < 1$ (irreducibility)

- $E[X] = \frac{\rho}{(1-\rho)} = \frac{\lambda^0}{\mu(1-p) - \lambda^0}$
- $T = \frac{1}{\mu(1-p) - \lambda^0}$
- Departure rate: $(1-\pi(0))\mu(1-p) = \rho\mu(1-p) = \lambda^0$
Ex. 2: Tandem open Jackson network

Traffic eqns:

\[ \lambda_1 = \lambda^0 + \lambda_K p \]
\[ \lambda_i = \lambda_{i-1} \text{ for } i = 2,..,K \]

- If \( p = 1 \) then no solution (non-irreducible)
- If \( p < 1 \) then \( \lambda_i = \lambda^0/(1-p) \) for \( i = 2,..,K \)
Ex. 2: Tandem open Jackson network

Limiting distribution $\pi(n) = \prod_{i=1}^{K} (1 - \rho_i) \rho_i^{n_i}$ for $n = (n_1, \ldots, n_K)$ provided $\rho_i := \frac{\lambda_0}{\mu_i(1-p)} < 1$ for $i=1,\ldots,K$ (stability) and $p<1$ (irreducibility)

- $T = K/(\mu(1-p)-\lambda^0)$
- Departure rate $= P(X_K>0)\mu_K(1-p) = \rho_K\mu_K(1-p) = \lambda^0$ as $P(X_K = 0) = \sum_{n_i\geq0, i=1,\ldots,K-1} \pi(n_1,\ldots,n_{K-1},0) = 1 - \rho_K$
Ex. 3: Open central server

\[ \lambda^0 \]

\[ p_1 + \ldots + p_K = 1 \]

Traffic eqns:
\[ \lambda_1 = \lambda^0 + \lambda_2 + \ldots + \lambda_K \]
\[ \lambda_i = \lambda_1 p_i \quad \text{for} \quad i = 2, \ldots, K \]

- If \( p_1 = 0 \) no solution
- If \( p_1 > 0 \) then \( \lambda_1 = \lambda^0 / p_1 \), \( \lambda_i = \lambda^0 p_i / p_1 \) for \( i = 2, \ldots, K \)
Ex. 3: Open central server (cont’)

Limiting distr. $\pi(n) = C \prod_{i=1}^{K} \rho_i^{n_i}$, $n = (n_1, ..., n_K)$
provided that $\rho_i := \frac{\lambda^0 p_i}{(\mu_i p_1)} < 1$ for $i = 1, ..., K$
(stability) and $p_1 > 0$ (irreducibility)

- $E[X_i] = \frac{\lambda^0 p_i}{(\mu_i p_1 - \lambda^0 p_i)}$ for $i = 1, ..., K$
- $T = \sum_{1 \leq i \leq K} \frac{p_i}{(\mu_i p_1 - \lambda^0 p_i)}$