More general arrival processes
Kleinrock’s conservation law 
(seen during Lecture 15)

- $W_i$ : expected waiting of class-$i$ in $M/G/1$ multiclass queue

**Result** (Kleinrock’s conservation law)

$\sum_{1 \leq i \leq K} \rho_i W_i$ does not depend on service policy and is equal to $R \rho / (1 - \rho)$ when $\rho < 1$ and $\infty$ when $\rho \geq 1$, with $R = (1/2) \sum_{1 \leq i \leq K} \lambda_i \sigma_i^{(2)}$ the expected residual load in server at arbitrary time (or at arrival times since PASTA holds here)

In particular, by Little, $\sum_{1 \leq i \leq K} N_i$ does not depend on service policy when all mean service times are equal
Sanity checks: Non-preemptive policy

- Single server, infinite waiting room
- K classes of jobs:
  - Class-k arrive according to Poisson process, rate $\lambda_k$
  - $G_k(x) =$ cd of service times of class-k jobs, mean $1/\mu_k$, 2nd order-moment $\sigma_k^{(2)}$
- Class-i has preemptive priority over class-j, if $i<j$ but preemption from server not allowed
- All above rvs mutually independent

$$\rho_i = \frac{\lambda_i}{\mu_i}, \quad \rho = \sum_{1 \leq i \leq K} \rho_i$$
Sanity checks: Non-preemptive policy (con’t)

Non-preemptive priority policy: (seen in Lecture 15)

\[
W_i = \frac{\sum_{j=1}^{K} \lambda_j \overline{\sigma}_j}{2 \left( 1 - \sum_{j=1}^{i-1} \rho_j \right) \left( 1 - \sum_{j=1}^{i} \rho_j \right)}
\]

\[
\sum_{i=1}^{K} \rho_i W_i = R \sum_{i=1}^{K} \frac{\rho_i}{\left( 1 - \sum_{j=1}^{i-1} \rho_j \right) \left( 1 - \sum_{j=1}^{i} \rho_j \right)}
\]

as \( R = (1/2) \sum_{j=1}^{K} \lambda_j \overline{\sigma}_j \)

Use induction argument to prove \( \sum_{1 \leq i \leq K} \rho_i W_i = R \rho / (1 - \rho) \)
\[
\sum_{i=1}^{K} \rho_i W_i = R \sum_{i=1}^{K} \frac{\rho_i}{\left(1 - \sum_{j=1}^{i-1} \rho_j\right) \left(1 - \sum_{j=1}^{i} \rho_j\right)} = R \frac{\rho}{1 - \rho} \quad \text{with } \rho = \sum_{1 \leq j \leq K} \rho_j
\]

True for \(K=1\) (\(M/G/1\) queue). Assume true up to \(K-1\)
Sanity checks: multiclass PS node

K classes, Poisson arrivals, rate $\lambda_i$ for class-i, generally distr. service times, mean $1/\mu_i$ for class-i

$X_i = \# \text{ class-i at equilibrium}$

- $W_i =$ expected waiting time class-i?

Showed in Lecture 12 that

$$P(X_1 = n_1, ..., X_K = n_K) = (1 - \rho)^n \prod_{i=1}^{K} \frac{\rho_i^{n_i}}{n_i!}$$

for $\rho = \sum_{1 \leq i \leq K} \rho_i < 1$, $\rho_i = \frac{\lambda_i}{\mu_i}$

Result insensitive w.r.t. service time distributions
Sanity checks: multiclass PS node (cont’)

Assume $\rho < 1$. We will use

$$\sum_{n_{i+\ldots+n_{K}=n}} n! \prod_{i=1}^{K} x_{i}^{n_{i}} / n_{i}! = \left( \sum_{i=1}^{K} x_{i} \right)^{n}$$

- $P(X_i = n) = ?$
- $F_i(z) = \sum_{n \geq 0} P(X_i = n) z^n$  \text{ z-transform of } X_i

$$= \sum_{n} z^{n_{i}} \pi(n) = (1 - \rho) \sum_{n} z^{n_{i}} \prod_{r=1}^{K} \rho_{r}^{n_{r}} / n_{r}!$$

$$= (1 - \rho) \sum_{n \geq 0} \sum_{n_{1}+\ldots+n_{K}=n} \prod_{r=1}^{i-1} \rho_{r}^{n_{r}} / n_{r}! \left( \prod_{r=1}^{i} \rho_{i} z^{n_{i}} / n_{i}! \right) \left( \prod_{r=i+1}^{K} \rho_{r}^{n_{r}} / n_{r}! \right)$$

$$= (1 - \rho) \sum_{n \geq 0} (\rho_1 + \ldots + \rho_{i-1} + \rho_i z + \rho_{i+1} + \ldots + \rho_K)^n$$

$$= \frac{1 - \rho}{1 - \rho + \rho_i + \rho_i z} \quad \text{as } \rho = \sum_{r=1}^{K} \rho_r$$
Since $1/(1-az) = \sum_{n\geq 0} (az)^n$ we can write $F_i(z)$ as

$$F_i(z) = \frac{1-\rho}{1-\rho + \rho_i + \rho_i z}$$

$$= \frac{1-\rho}{1-\rho + \rho_i} \frac{1}{1-az} \quad \text{with} \quad a = \rho_i z / (1-\rho + \rho_i)$$

to get (inversion of z-transform)

$$P(X_i=n) = \frac{1-\rho}{1-\rho + \rho_i} \left( \frac{\rho_i}{1-\rho + \rho_i} \right)^n$$

- $N_i$ = expected number of class-i customers

  $$= (d/dz)F_i(z)|_{z=1} = \frac{\rho_i}{(1-\rho)}$$

(or can be obtained from $P(X_i=n)$)
Sanity checks: multic和平 PS node (cont')

- $T_i = \text{expected sojourn of class-}i\text{ customers}$

  $$T_i = \frac{N_i}{\lambda_i} = \frac{1}{\mu_i(1 - \rho)} \quad \text{by Little}$$

- $W_i = \text{expected waiting time of class-}i\text{ customers}$

  $$W_i = T_i - \frac{1}{\mu_i} = \frac{\rho}{\mu_i(1 - \rho)}$$
Sanity checks: multiclass PS node (cont')

- LCFS/PS multiclass node:
  \[ W_i = \frac{\rho / \mu_i}{1 - \rho} \]

When all service times are exponential with mean $1/\mu_i$ for class-i:

\[ \sum_{i=1}^{K} \rho_i W_i = \frac{\rho}{1 - \rho} \sum_{i=1}^{K} \rho_i / \mu_i \]

\[ R = \frac{1}{2} \sum_{i=1}^{K} \lambda_i \sigma_i^{(2)} = \frac{1}{2} \sum_{i=1}^{K} \lambda_i \frac{2}{\mu_i^2} = \sum_{i=1}^{K} \rho_i / \mu_i \]

Therefore

\[ \sum_{i=1}^{K} \rho_i W_i = R \frac{\rho}{1 - \rho} \]
M/M/1 queue revisited

- One minor modification, arrival rate $\lambda'$ when no one in system; $\lambda$ otherwise
- Infinitesimal generator $Q$

\[
Q = \begin{bmatrix}
-\lambda' & \lambda' & 0 & 0 & \cdots \\
\mu & -(\lambda + \mu) & \lambda & 0 & \cdots \\
0 & \mu & -(\lambda + \mu) & \lambda & \cdots \\
0 & 0 & \mu & -(\lambda + \mu) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
- Balance equations

\[-\pi_0 \lambda' + \pi_1 \mu = 0\]
\[\pi_0 \lambda' - \pi_1 (\lambda + \mu) + \pi_2 \mu = 0\]
\[\pi_{j-1} \lambda - \pi_j (\lambda + \mu) + \pi_{j+1} \mu = 0, \quad j = 2, 3, \ldots\]

- Guess \(\pi_j = \pi_1 r^{j-1}\) for \(j \geq 2\) implies

\[\pi_1 r^j \mu - \pi_1 r^{j-1} (\lambda + \mu) + \pi_1 r^{j-2} \lambda = 0\]

or

\[r^2 \mu - r(\lambda + \mu) + \lambda = 0\]
Notice that there are two solutions to the last equation, \( r = \frac{\lambda}{\mu} \), 1

Solution \( r = 1 \) does not make sense since

\[
\sum_{j=1}^{\infty} \pi_j = \infty
\]

Therefore, when \( \frac{\lambda}{\mu} < 1 \), solution \( r = \frac{\lambda}{\mu} \) makes sense, and \( \pi_j = \pi_1 \rho^{j-1} \) (where \( \rho := \frac{\lambda}{\mu} \))
\( \pi_0 \) and \( \pi_1 \) determined with

\[
\pi_0 \lambda' = \pi_1 \mu \quad \text{and} \quad \sum_{j \geq 0} \pi_j = \pi_0 + \pi_1 \sum_{j \geq 1} \rho^{j-1} = 1
\]

Gives

\[
\pi_0 = \frac{\mu / \lambda'}{\mu / \lambda' + 1 / (1 - \rho)}
\]

\[
\pi_j = \frac{\rho^{j-1}}{\lambda' / \mu + 1 / (1 - \rho)} \quad \text{for } j \geq 1, \rho < 1
\]

\( \mathbb{E}[X] = ? \)

\[
= \sum_{j \geq 1} j \pi_j = \frac{1}{\lambda' / \mu + 1 / (1 - \rho)} \frac{1}{(1 - \rho)^2}
\]
Markov Modulated Poisson Process (MMPP)

- Given a K-state continuous-time MC \( \{X(t)\} \) with transition rates \( \{v_{i,j}\} \).

- Given K different arrival rates \( \{\lambda_k\} \)

- Arrival process is Poisson with rate \( \lambda_{X(t)}, t>0 \), i.e., the CTMC \( \{X(t), t\geq 0\} \) modulates the arrival rate

- Used to model bursty traffic sources in networks
MMPP/M/1 queue

- Arrivals according to a MMPP with transition rates $\{\nu_{i,j}\}$ and arrival rates $\{\lambda_k\}$

- Exponential service times, rate $\mu$

- System state: $(n,s)$ with $n = 0,1,...$ # of jobs in system and $s = 1,...,K$ state of CTMC modulating arrivals

- Defined a CTMC with state-space $\{0,1,...\} \times \{1,2\}$ with infinitesimal generator $Q$
MMPP/M/1 queue (cont’)

- Order state in lexicographic order
  \((0,1), \ldots, (0,K), (1,1), \ldots, (1,K), \ldots, (n,1), \ldots, (n,K), \ldots\)

- Transition from \((0,k)\) \(\rightarrow\) \((1,k)\) with rate \(\lambda_k\)
  \(\rightarrow\) \((0,j)\) with rate \(\nu_{k,j}\), \(k \neq j\)

- Transition from \((n,k)\) \(\rightarrow\) \((n+1,k)\) with rate \(\lambda_k\)
  \((n>0)\) \(\rightarrow\) \((n,j)\) with rate \(\nu_{k,j}\), \(k \neq j\)
  \(\rightarrow\) \((n-1,k)\) with rate \(\mu\)

- All the other transition rates are zero
MMPP/M/1 queue: two states (K=2)

\[
Q = \begin{bmatrix}
    B_{0,0} & A_0 & 0 & \ldots \\
    A_2 & A_1 & A_0 & 0 & \ldots \\
    0 & A_2 & A_1 & A_0 & \ldots \\
    0 & 0 & A_2 & A_1 & \ldots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

B_{00}, A_0, A_1, A_2

2-by-2 matrices

\[
B_{00} = \begin{pmatrix}
    -(\lambda_1 + \nu_{1,2}) & \nu_{1,2} \\
    \nu_{2,1} & -(\lambda_2 + \nu_{2,1})
\end{pmatrix}
\]

\[
A_2 = \begin{pmatrix}
    \mu & 0 \\
    0 & \mu
\end{pmatrix} \quad A_1 = \begin{pmatrix}
    -(\lambda_1 + \nu_{1,2} + \mu) & \nu_{1,2} \\
    \nu_{2,1} & -(\lambda_2 + \nu_{2,1} + \mu)
\end{pmatrix} \quad A_0 = \begin{pmatrix}
    \lambda_1 & 0 \\
    0 & \lambda_2
\end{pmatrix}
\]
Let \( \pi_j = (\pi_{j1}, \pi_{j2}) \), \( j = 0,1, \ldots \).

**Interpretation:**

\( \pi_{jk} = P(j \text{ cust. & modulated process in state } k) \), \( k=1,2 \)

\[
Q = \begin{bmatrix}
B_{0,0} & A_0 & 0 & \cdots \\
A_2 & A_1 & A_0 & 0 & \cdots \\
0 & A_2 & A_1 & A_0 & \cdots \\
0 & 0 & A_2 & A_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

\( \pi Q = 0 \) (Bal. eqns) equivalent to

\[
\begin{align*}
\pi_0 B_{00} + \pi_1 A_2 &= 0 \\
\pi_{j-1} A_0 + \pi_j A_1 + \pi_{j+1} A_2 &= 0 \quad j \geq 1
\end{align*}
\]

**Conjecture existence of 2-by-2 matrix** \( R \) **such that**

\( \pi_j = \pi_0 R^j \), \( j \geq 1 \)
Plugging $\pi_j = \pi_0 R^j$, $j \geq 1$, into
$$\pi_{j-1} A_0 + \pi_j A_1 + \pi_{j+1} A_2 = 0 \quad j \geq 1$$
gives
$$0 = \pi_0 R^{j-1} A_0 + \pi_0 R^j A_1 + \pi_0 R^{j+1} A_2$$
$$= \pi_0 R^{j-1} (A_0 + RA_1 + R^2 A_2)$$

If true, then $A_0 + RA_1 + R^2 A_2 = 0$

Careful: $R$ is a matrix!
Solution of $A_0 + RA_1 + R^2 A_2 = 0$

- Can be shown that this matrix equation has two solutions:
  - the matrix with all entries equal to 1
  - a second matrix

If system is stable, second matrix is the correct solution and has spectral radius less than 1 (similar to $\rho < 1$ in M/M/1 queue) i.e., all eigenvalues are less than one

- Solution matrix called *rate matrix*
  
  ➔ Matrix geometric solution
To get $\pi_0$ use $1 = \pi_0 \sum_{j \geq 0} R_j.1 = \pi_0 (I-R)^{-1}.1$
along with $\pi_0 B_{0,0} + \pi_1 A_2 = 0$ where $\pi_1 = \pi_0 R$

- Expected # customers:
  $\text{prob. } j \text{ cust. }$
  $\sum_{j \geq 1} j (\pi_{j,1} + \pi_{j,2}) = \sum_{j \geq 1} j \pi_j.1 = \pi_0 R (I-R)^{-2}.1$

- Stability condition: the modulating CTMC must be stable with stationary distribution

$$\pi' = (\pi'_1, \pi'_2) \text{ such that } \sum_{k=1}^{2} \pi'_k \lambda_k < \mu$$
**MMPP/M/1 queue: K states (K≥2)**

For $K=2$ (reminder), we have the matrix $Q$ defined as:

$$Q = \begin{bmatrix}
B_{0,0} & A_0 & 0 & \cdots \\
A_2 & A_1 & A_0 & 0 & \cdots \\
0 & A_2 & A_1 & A_0 & \cdots \\
0 & 0 & A_2 & A_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

Where:

- $B_{0,0} = \begin{pmatrix} - (\lambda_1 + \nu_{1,2}) & \nu_{1,2} \\ \nu_{2,1} & - (\lambda_2 + \nu_{2,1}) \end{pmatrix}$

- $A_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$

- $A_1 = \begin{pmatrix} - (\lambda_1 + \nu_{1,2} + \mu) & \nu_{1,2} \\ \nu_{2,1} & - (\lambda_2 + \nu_{2,1} + \mu) \end{pmatrix}$

- $A_0 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
MMPP/M/1 queue: K states (cont’)

\textbf{K=2} (reminder)

\[ B_{00} = \begin{pmatrix} -\left(\lambda_1 + \nu_{1,2}\right) & \nu_{1,2} \\ \nu_{2,1} & -\left(\lambda_2 + \nu_{2,1}\right) \end{pmatrix} \]

\[ A_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}, \quad A_1 = \begin{pmatrix} -\left(\lambda_1 + \nu_{1,2} + \mu\right) & \nu_{1,2} \\ \nu_{2,1} & -\left(\lambda_2 + \nu_{2,1} + \mu\right) \end{pmatrix}, \quad A_0 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \]

\( A_0, A_1, A_2 \) can be rewritten as

\[ A_0 = \text{diag}(\lambda_1, \lambda_2) \]

\[ A_2 = \text{diag}(\mu, \mu) \]

\[ A_1 = B_{00} - A_2 \]
**MMPP/M/1 queue: K states (cont')**

\[ K \geq 2 \]

\[
Q = \begin{bmatrix}
B_{0,0} & A_0 & 0 & \cdots \\
A_2 & A_1 & A_0 & 0 & \cdots \\
0 & A_2 & A_1 & A_0 & \cdots \\
0 & 0 & A_2 & A_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

\[ B_{00}, A_0, A_1, A_2 \]

K-by-K matrices

\[
B_{00} = \begin{pmatrix}
-(\lambda_1 + \nu_{1,2} + \ldots + \nu_{1,K}) & \nu_{1,2} & \nu_{1,3} & \cdots & \nu_{1,K} \\
\nu_{2,1} & -(\lambda_2 + \nu_{2,1} + \ldots + \nu_{2,K}) & \nu_{2,3} & \cdots & \nu_{2,K} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\nu_{K,1} & \nu_{K,2} & \cdots & \cdots & (\lambda_K + \nu_{K,1} + \ldots + \nu_{K,K-1})
\end{pmatrix}
\]

\[ A_0 = diag(\lambda_1, \ldots, \lambda_K) \quad A_2 = diag(\mu, \ldots, \mu) \quad A_1 = B_{00} - A_2 \]
MMPP/M/1 queue: K states (cont')

Let \( \pi_j = (\pi_{j1}, \ldots, \pi_{jK}), \ j \geq 0 \)

**Interpretation:**

\( \pi_{jk} = P(\text{j cust. \& modulated process in state } k) \)

\( \pi Q = 0 \) (balance eqns) equivalent to

\[
\pi_0 B_{00} + \pi_1 A_2 = 0 \\
\pi_{j-1} A_0 + \pi_j A_1 + \pi_{j+1} A_2 = 0, \ j \geq 1
\]

Conjecture existence K-by-K matrix \( R \) such that

\( \pi_j = \pi_0 R^j, \ j \geq 1 \)

\[
A_0 + RA_1 + R^2 A_2 = 0 \rightarrow \text{Same as } K=2 \text{ analysis}
\]
How to solve $A_0 + RA_1 + R^2 A_2 = 0$?

Simple iterative scheme:

$$R_0 = 0$$

$$R_{n+1} = -(A_0 + R_n^2 A_2) A_1^{-1}$$

Stop when $||R_{n+1} - R_n||$ small enough

E.g. $||A-B|| = \sup_{1 \leq i,j \leq n} |a_{i,j} - b_{i,j}|$ when $A = [a_{i,j}]_{1 \leq i,j \leq n}$ $B = [b_{i,j}]_{1 \leq i,j \leq n}$

Number of iterations needed increases as spectral radius of $R$ increases or as utilization increases (similar to $\rho \rightarrow 1$ for $M/M/1$ queue)
MMPP/M/1 queue: K states (cont')

To get \( \pi_0 \) use
\[
1 = \pi_0 \sum_{j \geq 0} R^j.1 = \pi_0 (I-R)^{-1}.1
\]
and
\[
0 = \pi_0 B_{00} + \pi_1 A_2 = \pi_0 (B_{00} + RA_2)
\]
(1)
⇒ gives K+1 eqns for K unknowns \( \pi_{0,i} \), \( i=1,\ldots,K \)
(1 eq. in (1) is redundant like in scalar case)

- Expected # customers:
\[
\sum_{j \geq 1} j \pi_j.1 = \pi_0 \sum_{j \geq 1} j R^j.1 = \pi_0 R(I-R)^{-2}.1
\]

- Stability condition: the modulating CTMC must be stable with stationary distribution

\[
\pi' = (\pi'_1,\ldots,\pi'_K) \text{ such that } \sum_{k=1}^{K} \pi'_k \lambda_k < \mu
\]
Phase-type (PH) distribution

Absorbing CTMC with transient states 1,...,N, absorbing state 0, infinitesimal generator

\[ Q = \begin{pmatrix} A & R \\ 0 & 0 \end{pmatrix} \]

A = \([q_{ij}]_{1 \leq i,j \leq N}\) N-by-N matrix, 
R = \([q_{i,0}]_{1 \leq i \leq N}\) column vector

\[ X = PH(a,A) \text{ with } a=(a_1,...,a_N), \]
\[ a_i = \text{prob. starting state is } i \]

- \[ P(X<x) = 1 - ae^{-Ax}1 \] cdf of X
- \[ E[X^n] = (-1)^n n!aA^{-n}1 \] n=1,2,...
Phase-type (PH) distribution (cont’)

Examples of PH distributions:

- **Exponential rv** \( X \) with rate \( \lambda \)
  \[ X = \text{PH}(a,A) \text{ with } a=(1), A=\begin{pmatrix} -\lambda \end{pmatrix} \]

- **Hyperexponential rv** \( X \): with prob. \( a_i/a \) with 
  \[ a := a_1 + \ldots + a_N \] 
  \( X \) is an exponential rv with rate \( \lambda_i \)
  \[ X = \text{PH}(a,A) \text{ with } a=(a_1,\ldots,a_N), A=\text{diag}(-\lambda_1,\ldots,-\lambda_N) \]

- **N-stage Erlang rv**: \( X = \text{PH}(a,A) \text{ with } a=(1,0,\ldots,0) \)

  \[ A = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & 0 & 0 \\ 0 & 0 & -\lambda & \lambda & 0 \\ 0 & 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & 0 & -\lambda \end{bmatrix} \]

  \( A = \text{N-by-N matrix} \)

  \( a = \text{vector of size } N \)
**Coxian rv:** there are $K$ exp. stages $1, 2, \ldots, K$

$$X = PH(a, A) \text{ with } a = (1, 0, \ldots, 0) \text{ and }$$

$$A = \begin{bmatrix}
-\lambda_1 & p_1 \lambda_1 & 0 & \cdots & 0 & 0 \\
0 & -\lambda_2 & p_2 \lambda_2 & \cdots & 0 & 0 \\
0 & 0 & -\lambda_3 & p_3 \lambda_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -\lambda_{k-2} & p_{k-2} \lambda_{k-2} & 0 \\
0 & 0 & \cdots & 0 & -\lambda_{k-1} & p_{k-1} \lambda_{k-1} \\
0 & 0 & \cdots & 0 & 0 & -\lambda_k 
\end{bmatrix}$$
Phase-type (PH) distribution (cont’)

**Result:** Any distribution can be arbitrarily well approximated by a phase-type distribution

So, at least in theory, one can approximate any GI/GI/1 queue* by a PH/PH/1 queue

* both interarrival times and service times sequences are i.i.d. and both sequences are independent (in particular, arrivals are not correlated unlike with MMPP arrival process)
PH/PH/1 queue

Q: How can we construct a PH/PH/1 queue?

A: Take two independent absorbing CTMCs with infinitesimal generators $Q_a$ and $Q_s$

- each time an absorption occurs in the CTMC with infinitesimal generator $Q_a$, a customer joins the queue
- each time an absorption occurs in the CTMC with infinitesimal generator $Q_s$, a customer leaves the server if the queue was non-empty
**PH/PH/1 queue**

Markov representation of PH/PH/1 queue: $(n,i,j)$ with

- $n = \text{nb. customers in queue}$
- $i = \text{state of absorbing CTMC with } Q_a = [a_{ij}]$
- $j = \text{state of absorbing CTMC with } Q_s = [s_{ij}]$

Assume for sake of simplicity that both CTMCs restart in state 1 (both have absorbing state 0)

- Non-zero transition rates from state $(n,i,j)$:
  
  $(n,i,j) \rightarrow (n-1,i,1)$ rate $s_{i0}1(n>0)$
  
  $\rightarrow (n+1,1,j)$ rate $a_{i0}$
  
  $\rightarrow (n,i',j)$ rate $a_{ii'}$
  
  $\rightarrow (n,i,j')$ rate $s_{jj'}$
Markov representation of queue has generator infinitesimal of the form

\[
Q = \begin{pmatrix}
B_0 & A_0 & 0 & 0 & 0 & \cdots \\
B_1 & A_1 & A_0 & 0 & 0 & \cdots \\
B_2 & A_2 & A_1 & A_0 & 0 & \cdots \\
B_3 & A_3 & A_2 & A_1 & A_0 & \cdots \\
B_4 & A_4 & A_3 & A_2 & A_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
More general structures (cont’)

Theorem 1.7.1. The irreducible Markov process $\tilde{Q}$ is positive recurrent if and only if the minimal nonnegative solution $R$ of the equation

$$\sum_{k=0}^{\infty} R^k A_k = 0 \quad (1.7.7)$$

has $\text{sp}(R) < 1$, and if there exists a positive vector $x_0$ such that

$$x_0 B[R] = 0. \quad (1.7.8)$$

The matrix $B[R] = \sum_{k=0}^{\infty} R^k B_k$ is a generator.

The stationary probability vector $x$, satisfying $x\tilde{Q} = 0$, $xe = 1$, is then given by

$$x_k = x_0 R^k, \quad \text{for} \quad k \geq 0, \quad (1.7.9)$$

and $x_0$ is normalized by

$$x_0 (I - R)^{-1}e = 1. \quad (1.7.10)$$