Stochastic Processes

Part 1
690PE_16 Class 2
Stochastic process $X = \{X(t), \, t \in T\}$ - collection of random variables (rvs)

- one rv for each $X(t)$ for each $t \in T$
- index set $T$ - possible set of values of $t$
- state space - possible set of values of $X(t)$
- if $T$ is countable, then $X$ is discrete-time process, will use notation $X = \{X_n, \, n \in \mathbb{N}\}$
- if $T$ is continuous, then $X = \{X(t), \, t \in T\}$ is continuous-time process
- $X(t)$ can take values from discrete- or continuous-state space
Examples

- nb. transactions processed by database system during interval \((0, t)\) - cont.-time, discrete-space
- nb. packets thru router during n-th hour of day, \(\{X_n, n = 1, 2, \ldots, 24\}\), discrete-time, discrete-space
- response time of request to google server given that it arrives at time \(t\), \(\{X(t), t > 0\}\), cont.-time, continuous space
- **Bernoulli process**: \(\{Y_n, n = 0, 1, \ldots\}\),

\[
P(Y_n = i) = \begin{cases} 
p, & i = 0, \\
1 - p, & i = 1, \\
0, & i \neq 0, 1. \\
\end{cases}
\]
Counting Process

*Counting process*: stochastic process that represents nb. of events that occurs by time \( t \). It’s a continuous-time, discrete-state (integers) process \( \{N(t), \ t \geq 0\} \)

**Definition**: \( \{N(t), \ t \geq 0\} \) is a counting process if

- \( N(0) = 0 \)
- \( N(t) \geq 0 \)
- \( N(t) \) increasing (nondecreasing) in \( t \)
- \( N(t) - N(s) \) is nb. events in interval \([s,t] \)
Bernoulli process

- \( N_i \) - nb. successes by time \( i = 0,1,... \)

\( N = \{N_i \mid i=0,1,\ldots\} \) counting process with independent, stationary increments

- \( p \) - failure prob.
- \( (1-p) \) - success probability

\[
P(N_i = n) = \binom{i}{n} (1-p)^n p^{i-n}, \quad n = 0,1,\ldots,i
\]

where by convention \( P(N_0 = 0) = 1 \)
Bernoulli process (cont')
Bernoulli process (cont’)

\[ P(N_i = n) = \binom{i}{n}(1-p)^n p^{i-n} \]

\( x = \text{success} \)

\( n, (1-p)^n \)

\( i-n, p^{i-n} \)
Bernoulli process

- \( N_i \) - no. successes by time \( i = 0,1,... \)

\( N = \{N_i \mid i=0,1,\ldots\} \) with independent, stationary increments

- \( p \) - failure prob. \( (1-p) \) success probability

\[ P(N_i = n) = \binom{i}{n} (1-p)^n p^{i-n}, \quad n = 0,1,...,i \]

\[ E[N_i] = i(1-p), \quad \sigma^2_{N_i} = ip (1 - p), \quad i = 0,1,... \]

- \( X \) - time between two consecutive successes
Bernoulli process (cont')

\[ P(X=n) = p^{n-1}(1-p) \quad n = 1,2,... \]
Bernoulli process (cont’)

- \( N_i \) - nb. successes by time \( i = 0,1,... \)

\[ N = \{ N_i \ i = 0,1,.. \} \] with independent, stationary increments

- \( p \) - failure probability \((1-p)\) - success prob.

\[ P(N_i = n) = \binom{i}{n} (1-p)^n p^{i-n}, \ n = 0,1,...,i \]

\[ E[N_i] = i(1-p), \ \sigma^2_{N_i} = ip (1 - p), \ i = 0,1,... \]

- \( X \) - time between successes

\[ P(X = n) = (1-p)p^{n-1}, \ n = 1,2,... \]

\[ E[X] = 1/(1-p), \ \sigma^2_X = p/(1-p)^2 \]

\[ F_X(n) = P(X \leq n) = 1 - p^n, \ n = 1,2,... \]
Bernoulli process (cont’)

- \(X^{(n)}\) - time between success and \(n\)-th successive success

\[
P(X^{(n)} = k) = \binom{k-1}{n-1} (1 - p)^n p^{k-n}, \ k = n, n+1, \ldots
\]

called negative binomial distribution

*Depend on two parameters, \(n\) and \(k\)*
Bernoulli process (cont')

\[ P(X^{(n)} = k) = P(N_{n-1} = k-1)(1 - p), \quad k = n, n+1, \ldots \]

\[ = \binom{k - 1}{n - 1} (1 - p)^{n-1} p^{k-n} (1-p) \]

\[ = \binom{k - 1}{n - 1} (1 - p)^{n} p^{k-n} \]
Bernoulli process (cont’)

- $X^{(n)}$ - time between success and $n$-th successive success,

  \[ P(X^{(n)} = k) = \binom{k-1}{n-1} (1 - p)^n p^{k-n}, \quad k = n, n+1, \ldots \]

  called negative binomial distribution

  \[ E[X^{(n)}] = \frac{n}{1 - p} \]

Memoryless property

\[ P(X = l + n \mid X > l) = (1 - p)p^{n-1}, \quad l > 0; n \geq 1 \]
Little o notation

Definition: $f$ is $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

- $f(h) = h^2$ is $o(h)$
- $f(h) = h$ is not
- $f(h) = h^r$, $r > 1$ is $o(h)$
- $\sin(h)$ is not $(\sin(h)/h \to 1)$
- If $f, g$ are $o(h)$, then $f(h) + g(h) = o(h)$
- If $f$ is $o(h)$, then $c \times f(h) = o(h)$, $|c| < \infty$
Example: exponential rv $X$ with parameter $\lambda$
has distribution $P(X < h) = 1 - e^{-\lambda h}$, $h > 0$

\[
P(X \leq t + h \mid X > t) = P(X \leq h) = 1 - e^{-\lambda h} \\
= 1 - [1 - \lambda h + \sum_{n=2}^{\infty} (-\lambda h)^n / n!] \\
= \lambda h + o(h)
\]
Poisson process

Counting process \( \{N(t), \ t \geq 0\} \) with rate \( \lambda > 0 \)

1. independent and stationary increments
2. \( P(N(h) = 1) = \lambda h + o(h) \)
3. \( P(N(h) \geq 2) = o(h) \)

(2)-(3) implies \( P(N(h) = 0) = (1- \lambda h) + o(h) \)

Let \( P_n(t) = P(N(t) = n) \)
Solution

\[ P_0(t+h) = P(N(t+h)=0) = P(N(t+h)=0|N(t)=0)P(N(t)=0) = P(N(h)=0)P_0(t) = (1- \lambda h + o(h)) P_0(t) = P_0(t) - \lambda h P_0(t) + o(h) \]

Hence \( (P_0(t+h)-P_0(t))/h = - \frac{\lambda P_0(t)}{h} + o(h)/h \)

\[ dP_0(t)/dt = -\lambda P_0(t) \quad \text{when } h \text{ goes to } 0 \]

yields \( P_0(t) = e^{-\lambda t} \) (with \( P_0(0)=1 \))
Solution (cont’)

\[ n > 0 \quad P_n(t+h) = P(N(t+h) = n) \]
\[ = P(N(t+h) = n | N(t) = n)P(N(t) = n) \]
\[ + P(N(t+h) = n | N(t) = n-1)P(N(t) = n-1) + o(h) \]
\[ = P(N(h) = 0) P_n(t) + P(N(h) = 1) P_{n-1}(t) + o(h) \]
\[ = (1 - \lambda h + o(h)) P_n(t) + (\lambda h + o(h)) P_{n-1}(t) + o(h) \]
\[ = P_n(t) - \lambda h P_n(t) + \lambda h P_{n-1}(t) + o(h) \]

as \( o(h)P_n(t) + o(h)P_{n-1}(t) + o(h) = o(h) \)

Hence \( (P_n(t+h) - P_n(t))/h = -\lambda P_n(t) + \lambda P_{n-1}(t) + o(h)/h \)

\[ dP_n(t)/dt = -\lambda P_n(t) + \lambda P_{n-1}(t) \]

yields \( P_n(t) = e^{-\lambda t} (\lambda t)^n / n! \)

by induction (with \( P_n(0) = 1 \) for \( n > 0 \))
Poisson process (cont')

Counting process \( \{N(t), t \geq 0\} \) with rate \( \lambda > 0 \)

1. independent and stationary increments

2. \( P(N(h) = 1) = \lambda h + o(h) \)

3. \( P(N(h) \geq 2) = o(h) \)

\[ P(N(h) = 0) = (1 - \lambda h) + o(h) \]

Let \( P_n(t) = P(N(t) = n) \)

\[ P_n(t) = e^{-\lambda t} (\lambda t)^n / n! \], \( n = 0, 1, ... \)

\[ E[N(t)] = \lambda t, \quad \sigma^2_{N(t)} = \lambda t \]

\( X \) = time between two consecutive events

\( F_X(t) = P(X < t) \) (cdf), \( f_X(t) = dF_X(t) / dt \) (pdf)
Poisson process (cont’)

\[ P(X > t) = P(\text{no event in } [0, t)) = P_0 (t) = e^{-\lambda t} \]

Therefore \( P(X < t) = 1 - e^{-\lambda t} \)

\[ \Rightarrow X \text{ exponential rv with parameter } \lambda \]

\[ f_X (t) = - \lambda e^{-\lambda t} \]
Poisson process (cont')

Counting process \{N(t), t \geq 0\} with rate \(\lambda > 0\)

1. independent and stationary increments
2. \(P(N(h) = 1) = \lambda h + o(h)\)
3. \(P(N(h) > 2) = o(h)\)
   \(\Rightarrow P(N(h) = 0) = (1 - \lambda h) + o(h)\)

Let \(P_n(t) = P(N(t) = n)\)

\[ P_n(t) = e^{-\lambda t} (\lambda t)^n / n! \], \(n = 0,1,...\)

\[ E[N(t)] = \lambda t, \quad \sigma^2_{N(t)} = \lambda t \]

\(X\) = time between two consecutive events

\[ F_X(t) = 1 - e^{-\lambda t} \quad (\text{cdf}), \quad f_X(t) = -\lambda e^{-\lambda t} \quad (\text{pdf}) \]

\[ E[X] = 1/\lambda, \quad \sigma^2_X = 1/\lambda^2 \]
Poisson process (cont’)

- $X^{(n)}$, time from event until $n$-th successive event

sum of $n$ exponential and independent rvs
= Erlang distribution of order $n$

$$f_{X^{(n)}}(t) = e^{-\lambda t} \lambda (\lambda t)^{n-1}/(n-1)! \quad \text{pdf of } X^{(n)}$$
Poisson process (cont')

Proof by calculation that \( f_X^{(n)}(t) = e^{-\lambda t} \frac{\lambda (\lambda t)^{n-1}}{(n-1)!} \)

\[ P(X^{(n)} > t) = \text{Prob. (strictly less than } n \text{ events in } [0,t)) \]
\[ = \sum_{0 \leq i \leq n-1} P_i(t) = \sum_{0 \leq i \leq n-1} e^{-\lambda t} \frac{(\lambda t)^i}{i!} \]

\[ f_X^{(n)}(t) = \frac{dP(X^{(n)} < t)}{dt} = - \frac{dP(X^{(n)} > t)}{dt} \]
\[ = e^{-\lambda t} \frac{\lambda}{i!} \sum_{0 \leq i \leq n-1} (\lambda t)^i \]
\[ - e^{-\lambda t} \frac{\lambda}{(i-1)!} \sum_{1 \leq i \leq n-1} (\lambda t)^{i-1} \]
\[ = e^{-\lambda t} \frac{\lambda (\lambda t)^{n-1}}{(n-1)!} \]

as only 1\(^{st}\) term in 1\(^{st}\) sum remains, others cancel out
Poisson process: alternative definition

Take iid (independent identically distributed) rvs \{X_i\}_i with rate \lambda

Define \( N(t) = \max\{n \mid X_1 + \ldots + X_n \leq t\} \)

\{N(t), t>0\} is a Poisson process
Poisson process (cont’) 

- If $N(t)$ is Poisson process and one event occurs in $[0, t]$, then time to event is uniformly distributed in $[0, t]$, 
  
  \[ f_{X|N(t)=1}(x|1) = \frac{1}{t}, \quad 0 \leq x \leq t \]

- if $N_1(t), N_2(t)$ are independent Poisson processes with rates $\lambda_1, \lambda_2$, then $N(t) = N_1(t) + N_2(t)$ is Poisson process with rate $\lambda = \lambda_1 + \lambda_2$
Proof uniform distribution

$X, Y$ independent exponential rvs with rate $\lambda$

$Z = \text{arrival time of event in } (0,t) \text{ given only one event in } (0,t)$

$P(Z < u) = P(X < u | 1 \text{ event in } (0,t))$

$= P(X < u, 1 \text{ event in } (0,t))/P(1 \text{ event in } (0,t))$

$= P(X < u, Y > t - X)/P(1 \text{ event in } (0,t))$

$= \int_0^u P(Y > t - s) \lambda e^{-\lambda s} ds$

$= \frac{\int_0^u e^{-\lambda(t-s)} \lambda e^{-\lambda s} ds}{\lambda te^{-\lambda t}}$

$= \frac{\lambda ue^{-\lambda t}}{\lambda te^{-\lambda t}} = \frac{u}{t}$

$\Rightarrow f_Z(u) = 1/t \quad - \text{Uniform distribution in } (0,t)$
Poisson process (cont’)

- \( N(t) \) is Poisson with rate \( \lambda \), \( M_i \) is Bernoulli with success prob. \( p \). Construct new process \( L(t) \) by only counting \( n \)-th event in \( N(t) \) whenever \( M_n = 1 \) (i.e., success at time \( n \))

then \( L(t) \) is Poisson with rate \( \lambda p \)

- Exhibits memoryless property,

\[
\mathbb{P}(X > t \mid X > t) = \frac{\mathbb{P}(X > x \cap X > t)}{\mathbb{P}(X > t)} = \frac{\lambda e^{-\lambda(x-t)}}{\lambda e^{-\lambda t}} = e^{-\lambda t}
\]

or, if \( X = t+Y \), i.e., \( Y \) is the remaining time until event, then

\[
f_Y(y) = \lambda e^{-\lambda y} = f_X(y)
\]
Example

Consider a web server where failures are described by a Poisson process with rate $\lambda = 2.4/\text{day}$, i.e., the time between failures, $X$, is exponential rv with mean $E[X] = 10$ hrs.

- $P(\text{time between failures} < T \text{ days}) =$
- $P(\text{k failures in } T \text{ days}) =$
- $P(\text{N}(5) < 10) =$
- Look in on system at random day, what is prob. of no. failures during next day?
- Failure is memory failure with prob. $1/9$, CPU failure with prob. $8/9$. Failures occur as independent events. What is process governing memory failures?
Review

- Bernoulli process \( \{N_i\}_i \) with parameter \( p \) (prob. of event)
- Counting process with stationary and independent increments
  \[ P(N_i = n) = \binom{i}{n} p^n (1-p)^{i-n} \]
- \( X \) – interevent time (interarrival time)
  \[ P(X = k) = (1-p)^{k-1} p \]
- \( X^{(n)} \) – time to \( n \)-th successive event
Poisson process \{N(t), t>0\} with parameter \(\lambda\) (event rate)

- counting process with stationary and independent increments
  \[P(N(t) = n) = (\lambda t)^n e^{-\lambda t}/n!, \quad n=0,1,...\]

- \(X\) - interevent time (interarrival time)
  \[f_X(x) = \lambda e^{-\lambda t}, \quad t \geq 0\]

- \(X^{(n)}\) - time to \(n\)-th successive event
  \[f_{X^{(n)}}(t)=e^{-\lambda t} \lambda (\lambda t)^{n-1}/(n-1)! , \quad t \geq 0\]
Properties

- Bernoulli and Poisson processes exhibit memoryless properties, i.e.,
  \[ F_{X|X>t}(x-t) = F_X(x-t) \]

- sum of two independent Poisson processes with rates \( \lambda_1, \lambda_2 \) is Poisson with rate \( \lambda_1 + \lambda_2 \)

- thinning a Poisson process with rate \( \lambda \) using independent Bernoulli process with probability \( p \) yields Poisson process with rate \( \lambda p \)