Fluid modeling (con’t)

690PE_16 Class 20
Motivation

Often systems are too « big » to allow for exact or even approximate microscopic analysis

Call for macroscopic analysis to capture main system features/parameters/performance

A - Mean-field approach (Lecture 18)
B - Poisson-driven stochastic differential equation (Lecture 19)
C - Ad hoc fluid models - sequel of Lecture 19
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Call for macroscopic analysis to capture main system features/parameters/performance

A - Mean-field approach (Lecture 18)
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C - Ad hoc fluid models - sequel of Lecture 19
Example 1: \( \{X(t)\}_{t \geq 0} \) two-state CTMC with generator

\[
Q = \begin{pmatrix}
-\alpha & \alpha \\
\beta & -\beta \\
\end{pmatrix}
\]

and \( \lambda_i = \lambda \) if \( i=1 \) and \( \lambda_i = 0 \) if \( i=2 \)

\( \Rightarrow \) so-called « On-Off fluid source »

Q: \( \lim_{t \to \infty} P(C(t) \leq x) = ? \)

We assume that the reservoir is infinite
Example 1: Single On-Off fluid source

- $X_i$: length i-th On-period, exp. rv rate $\alpha$
- $Y_i$: length i-th Off-period, exp. rv rate $\beta$
- All rvs $\{X_i, Y_i\}_i$ are mutually independent

[Diagram showing input rate $\lambda$, buffer drained at rate $\mu$, and process $C(t)$]
Example 1 (cont'): \( \{X(t)\}_{t \geq 0} \) two-state CTMC, generator

\[
Q = \begin{pmatrix}
-\alpha & \alpha \\
\beta & -\beta
\end{pmatrix}
\]
and \( \lambda_i = 0 \) if \( i=1 \) and \( \lambda_i = \lambda > 0 \) if \( i=2 \)

\[\lim_{t \to \infty} P(X(t)=i) := P(X=i) = ? \quad i=1,2\]

Solving \( (P(X=1), P(X=2))Q = 0, P(X=1) + P(X=2) = 1 \) gives

\[
P(X = 1) = \frac{\beta}{\alpha + \beta}, \quad P(X = 2) = \frac{\alpha}{\alpha + \beta}
\]
Q: \( \lim_{t \to \infty} P(C(t) \leq x) = ? \) \( F_i(t,x) = P(C(t) \leq x, X(t) = i) \)

Assume \( \lambda > \mu \) (otherwise \( C(t) = 0 \) for all if \( C(0) = 0 \))

\[
F_1(t+h,x) = P(C(t+h) \leq x, X(t+h) = \text{On}) = \\
P(C(t) \leq x, X(t) = \text{Off}) \cdot P(\text{duration Off period} < h) \\
+ P(C(t) \leq x + (\mu - \lambda)h, X(t) = \text{On}) \cdot P(\text{duration On period} > h) + o(h) \\
= F_2(t,x) \beta h + F_1(t,x + (\mu - \lambda)h)(1 - \alpha h) + o(h)
\]

\[
F_1(t+h,x) - F_1(t,x) + F_1(t,x) - F_1(t,x-(\lambda-\mu)h) \\
= \beta h F_2(t,x) - \alpha h F_1(t,x + (\mu - \lambda)h)
\]

\[
[F_1(t+h,x) - F_1(t,x)]/h + (\lambda - \mu)[F_1(t,x) - F_1(t,x-(\lambda-\mu)h)]/(\lambda - \mu)h \\
= \beta F_2(t,x) - \alpha F_1(t,x + (\mu - \lambda)h)
\]

\( h \to 0 : \quad dF_1(t,x)/dt + (\lambda - \mu)dF_1(t,x)/dx = \beta F_2(t,x) - \alpha F_1(t,x) \)
\[ \frac{dF_1(t,x)}{dt} + (\lambda - \mu)\frac{dF_1(t,x)}{dx} = \beta F_2(t,x) - \alpha F_1(t,x) \]

Similarly
\[ F_2(t+h,x) = P(C(t+h) \leq x, X(t) = \text{Off}) = \alpha h F_1(t,x) + (1 - \beta h) F_2(t,x+h\mu) + o(h) \]

\[ \Rightarrow \frac{dF_2(t,x)}{dt} - \mu \frac{dF_2(t,x)}{dx} = \alpha F_1(t,x) - \beta F_2(t,x) \]
\[
\frac{dF_1(t,x)}{dt} + (\lambda - \mu) \frac{dF_1(t,x)}{dx} = \beta F_2(t,x) - \alpha F_1(t,x)
\]
\[
\frac{dF_2(t,x)}{dt} - \mu \frac{dF_2(t,x)}{dx} = \alpha F_1(t,x) - \beta F_2(t,x)
\]

At equilibrium (with \(\lim_{t \rightarrow \infty} F_i(t,x) := F_i(x)\))

\[
(\lambda - \mu) \frac{dF_1(x)}{dx} = \beta F_2(x) - \alpha F_1(x) \tag{4}
\]
\[
-\mu \frac{dF_2(x)}{dx} = \alpha F_1(x) - \beta F_2(x)
\]

Adding both sides \(\rightarrow (\lambda - \mu) \frac{dF_1(x)}{dx} - \mu \frac{dF_2(x)}{dx} = 0\)

so that \((\lambda - \mu)F_1(x) - \mu F_2(x) = c_0\) with \(c_0\) a constant

(4) becomes

\[
\frac{dF_1(x)}{dx} = \left( \frac{\beta}{\mu} - \frac{\alpha}{\lambda - \mu} \right) F_1(x) - \frac{\beta c_0}{(\lambda - \mu)\mu}
\]
\[
\frac{dF_1(x)}{dx} = aF_1(x) + b
\]
with \(a = \frac{\beta}{\mu} - \frac{\alpha}{(\lambda-\mu)}\), \(b = -\frac{\beta c_0}{\mu(\lambda-\mu)}\)

**Solution:** \(F_1(x) = c_1 e^{ax} - \frac{b}{a}\)

Stable if \(a < 0 \iff \frac{\beta}{\mu} < \frac{\alpha}{(\lambda-\mu)} \iff \frac{\beta \lambda}{(\alpha+\beta)} < \mu\)

Makes sense as \(P(X=1) = \frac{\beta}{(\alpha+\beta)}\)

\(c_0 = ?\)

Defined by \((\lambda-\mu)F_1(x) = \mu F_2(x) + c_0\) for all \(x \geq 0\)

\[(\lambda-\mu)F_1(\infty) = \mu F_2(\infty) + c_0\]

\[c_0 = (\lambda-\mu)P(X=1) - \mu P(X=2)\]

\[= \frac{((\lambda-\mu)\beta - \mu \alpha)}{(\alpha+\beta)}\]
\( F_1(x) = c_1 e^{ax} - b/a \) with \( a = \beta/\mu - \alpha/(\lambda-\mu) \),
\( b := -\beta c_0/(\mu(\lambda-\mu)) \) and \( c_0 = ((\lambda-\mu)\beta - \mu\alpha)/(\alpha+\beta) \)

\( c_1 = ? \)

\( F_1(0) = 0 \) implies \( c_1 = b/a \) yielding \( F_1(x) = -b/a(1-e^{ax}) \)

Moreover \( F_1(\infty) = P(X=1) = \beta/(\alpha+\beta) \) gives \( b/a = -\beta/(\alpha+\beta) \)

In summary

\[
F_1(x) = \frac{\beta}{\alpha + \beta} \left( 1 - e^\left( \frac{\beta-\alpha}{\mu(\lambda-\mu)}x \right) \right)
\]

\( F_2(x) \) obtained from \( (\lambda-\mu)F_1(x) = \mu F_2(x) + c_0 \)

\[
F_2(x) = \frac{\alpha}{\alpha + \beta} - \frac{\beta(\lambda - \mu)}{(\alpha + \beta)\mu} e^\left( \frac{\beta-\alpha}{\mu(\lambda-\mu)}x \right)
\]
\[
F_1(x) = \frac{\beta}{\alpha + \beta} \left( 1 - e^{\left( \frac{\beta - \alpha}{\mu - \lambda - \mu} \right)x} \right) \\
F_2(x) = \frac{\alpha}{\alpha + \beta} - \frac{\beta(\lambda - \mu)}{(\alpha + \beta)\mu} e^{\left( \frac{\beta - \alpha}{\mu - \lambda - \mu} \right)x} 
\]

**Q:** Average fluid level?

\[
\bar{C} = \int_0^\infty P(C > x) \, dx = \int_0^\infty \left( 1 - (F_1(x) + F_2(x)) \right) \, dx \\
= \frac{\beta \lambda (\lambda - \mu)}{\alpha + \beta} \left( \frac{1}{\mu(\lambda + \beta) - \lambda \beta} \right) 
\]
Let us come back to

\[(\lambda-\mu)dF_1(x)/dx = \beta F_2(x) - \alpha F_1(x)\]

\[-\mu dF_2(x)/dx = \alpha F_1(x) - \beta F_2(x)\]

Equivalent to

\[R \, d\overline{F}(x)/dx = Q^\top \, \overline{F}(x) \quad \text{with} \quad \overline{F}(x) = (F_1(x), F_2(x))^\top\]

\[R = \begin{pmatrix} \lambda - \mu & 0 \\ 0 & -\mu \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}\]

Since \(\lambda > \mu\) \(d\overline{F}(x)/dx = R^{-1}Q^\top \, \overline{F}(x)\)
Let us return to the general case

d\frac{C(t)}{dt}=0 \text{ if } C(t)=0 \text{ and } r_{X(t)} \leq 0 \text{ or if } C(t)=K \text{ and } r_{X(t)}>0
\therefore = r_{X(t)} \text{ otherwise}

with K finite or infinite (here } r_{x(t)}:=\lambda_{X(t)} - \mu \text{)

Nota: } r_i \text{ is the net input rate in state } i. \text{ It can be positive or negative or equal to zero

Q: } \lim_{t \to \infty} P(C(t) \leq x) = ?
\[ F(x) = (F_1(x), \ldots, F_N(x))^T \]
with \[ F_i(x) = \lim_{t \to \infty} P(C(t) \leq x, X(t) = i) \]
\[ R = \text{diag}(r_1, \ldots, r_N) \]
\[ Q = [q_{i,j}]_{1 \leq i,j \leq N} \] generator of CTMC \( \{X(t)\}_{t \geq 0} \)

Stability condition when reservoir is infinite is
\[ \Sigma_{1 \leq i \leq N} r_i p_i < 0 \]
with \( p = (p_1, \ldots, p_N) \) stat. distr. of \( \{X(t)\}_{t \geq 0} \) (i.e. \( pQ = 0 \), \( p1 = 1 \))

Easy to show that \( RF'(x) = Q^T F(x) \)

If \( r_i \neq 0 \) then \( R^{-1} \) exists and \( F'(x) = R^{-1}Q^T F(x) \)

From now on assume \( r_i \neq 0 \) for all \( i = 1, 2, \ldots, N \)
\( F'(x) = R^{-1}Q^T F(x) \)

- When eigenvalues of \( R^{-1}Q^T \) are simple (i.e. they are all different)

\[
F(x) = \sum_{j=1}^{N} c_j e^{\xi_j x} v^{(j)}
\]

with

- \((\xi_j, v^{(j)})\) eigenvalue-eigenvector pairs of \( R^{-1}Q^T \)
- \(c_1, \ldots, c_N\) constants found from boundary conditions (see next)
Check that \( \overline{F}(x) = \sum_{j=1}^{N} c_j e^{\zeta_j x} \overline{v}^{(j)} \) solves \( \overline{F}'(x) = R^{-1}Q^T \overline{F}(x) \)

\[
\overline{F}'(x) = \sum_{j=1}^{N} c_j e^{\zeta_j x} \zeta_j \overline{v}^{(j)} \\
= \sum_{j=1}^{N} c_j e^{\zeta_j x} R^{-1}Q^T \overline{v}^{(j)} \quad \text{as} \quad R^{-1}Q^T \overline{v}^{(j)} = \xi_j \overline{v}^{(j)} \\
= R^{-1}Q^T \sum_{j=1}^{N} c_j e^{\zeta_j x} \overline{v}^{(j)} \\
= R^{-1}Q^T \overline{F}(x)
\]
Assume $K$ is infinite from now on

$N_+ = \{i : r_i > 0\}, \ N_- = \{i : r_i < 0\}$

Note $N_+ \cup N_- = \{1, \ldots, N\}$ as $r_i \neq 0$ for all $i$

A remarkable result:

Under stability condition $\Sigma_{1 \leq i \leq N} r_i p_i < 0$

- nb. of eigenvalues with negative real part is $|N_+|$
- nb. of eigenvalues with positive real part is $|N_-|-1$
- 1 eigenvalue is zero

Rewrite as

$$F(x) = c_1 v^{(1)} + \sum_{j=2}^{\lfloor N+1 \rfloor} c_j e^{\xi_j x} v^{(j)} + \sum_{j=\lfloor N_+ \rfloor+2}^{N} c_j e^{\xi_j x} v^{(j)}$$

with

$\text{Re}(\xi_j) < 0$ for $j=2, \ldots, |N_+|+1$, $\text{Re}(\xi_j) > 0$ for $j=|N_+|+2, \ldots, N$
\[ F(x) = c_1 v^{(1)} + \sum_{j=2}^{\lfloor N_+ \rfloor + 1} c_j e^{\xi_j x} v^{(j)} + \sum_{j=\lfloor N_+ \rfloor + 2}^{N} c_j e^{\xi_j x} v^{(j)} \]

with \( \text{Re}(\xi_j) < 0 \) for \( j = 2, \ldots, \lfloor N_+ \rfloor + 1 \), \( \text{Re}(\xi_j) > 0 \) for \( j = \lfloor N_+ \rfloor + 2, \ldots, N \)

One must have \( c_j = 0 \) for \( j = \lfloor N_+ \rfloor + 2, \ldots, N \) as otherwise \( F(\infty) \) unbounded which is impossible as \( F(\infty) = p \)

Therefore
\[ F(x) = c_1 v^{(1)} + \sum_{j=2}^{\lfloor N_+ \rfloor + 1} c_j e^{\xi_j x} v^{(j)} \]

with \( \text{Re}(\xi_j) < 0 \) for \( j = 2, \ldots, \lfloor N_+ \rfloor + 1 \)
Determining constants $c_j$'s when $K$ is infinite

- $F(\infty) = \underline{p}$ with $\underline{p} = (p_1, \ldots, p_N)$ stationary distribution of CTMC modulating input rates

  Gives $c_1 \underline{v}^{(1)} = \underline{p}$ and

  \[
  F(x) = \underline{p} + \sum_{j=2}^{\lfloor N_+ \rfloor + 1} c_j e^{\varsigma_j x} \underline{v}^{(j)}
  \]

- $F_i(0) = 0$ when $r_i > 0$ (buffer cannot be empty in steady-state if net input rate strictly positive)
  \[ \rightarrow \text{there are exactly } N_+ \text{ states i s.t. } F_i(0) = 0 \]

  Yields linear system of $N_+$ equations

  \[
  A \underline{c} = \underline{b}^T \text{ with } \underline{c} = (c_j, j = 2, \ldots, \lfloor N_+ \rfloor + 1)^T \]

  \[
  \underline{b} = (p_j, j \in N_+)^T
  \]
\( W = \text{Average amount of fluid} \)

\[
W = \int_0^\infty \left( 1 - \sum_{i=1}^{N} F_i(x) \right) dx
\]
Example 2: The celebrated Anick et al. model

Superposition of N independent, statistically identical on-off fluid sources, infinite buffer

- $X^1_i$: length of the $i$th On-period source $n$, exp. rate $\alpha$
- $Y^1_j$: length of the $j$th Off-period source $n$, exp. rate $\beta$
- All rvs $\{X^m_i, Y^m_j\}_{m,n,i,j}$ are mutually independent

Buffer drained at rate $\tau < N$
Example 2: The celebrated Anick et al. model (cont’)

- Each source has generator $\begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$
- For each source, arrival rate in state $\text{On}$ is $1$, arrival rate in state $\text{Off}$ is $0$
- In steady-state, each source in state $\text{On/Off}$ with prob. $\frac{\beta}{\alpha + \beta}$, $\frac{\alpha}{\alpha + \beta}$, resp.

Let $N(t) = \# \text{sources in state } \text{On} \text{ at time } t$

$\pi_n := \lim_{t \to \infty} P(N(t)=n) = \binom{N}{n} \beta^n \alpha^{N-n} / (\alpha + \beta)^N$
Example 2: The celebrated Anick et al. model (cont’)

\[ \pi_n = \binom{N}{n} \beta^n \alpha^{N-n} / (\alpha + \beta)^N \] prob. n sources On

- Expected number of sources in state On

\[ \sum_{n=0}^{N} n \pi_n = \frac{1}{(\alpha + \beta)^N} \sum_{n=0}^{N} n \binom{N}{n} \beta^n \alpha^{N-n} = \frac{N \beta}{(\alpha + \beta)} \]

- System stable if \( \frac{N \beta}{(\alpha + \beta)} < r \)
Example 2: The celebrated Anick et al. model (cont’)

Can be cast into previous framework, as superposition of \( N \) independent CTMCs is CTMC

\[ N(t) = \# \text{ source in state } \text{On} \text{ at time } t \]

\( \{N(t)\}_{t \geq 0} \) is a CTMC with non-zero transition rates

\[ i \rightarrow i+1 \text{ with rate } (N-i)\beta \]

\[ i \rightarrow i-1 \text{ with rate } i\alpha \]

Previous theory applies to CTMC \( \{(C(t),N(t))\}_{t \geq 0} \)

with \( C(t) \) amount of fluid in buffer at time \( t \)
Example 2: The celebrated Anick et al. model (cont’)

Previous theory applies to CTMC \(\{(C(t), N(t))\}_{t \geq 0}\) with \(C(t)\) amount of fluid in buffer at time \(t\)

Define:

\[
F_n(x) = \lim_{t \to \infty} P(C(t) < x, N(t) = n)
\]

\[
F(x) = (F_1(x), \ldots, F_N(x))^T
\]

We get

\[
F(x) = \sum_{j=1}^{N} c_j e^{\xi_j x} \nu^{(j)}
\]

with

- \((\xi_j, \nu^{(j)})\) eigenvalue-eigenvector pairs of \(R^{-1}Q^T\)
Example 2: The celebrated Anick et al. model (cont’)

\[ F(x) = \sum_{j=1}^{N} c_j e^{s_j x} \phi^{(j)} \text{ with } R^{-1}Q^T \phi^{(j)} = \xi_j \phi^{(j)}, j=1,...,N \]

Here \( R = \text{diag}(-r, 1-r, ..., N-r) \) (take \( r \) not integer) and

\[ Q = \begin{pmatrix}
-\beta N & \beta N \\
\alpha & -(\alpha + (N-1)\beta) & (N-1)\beta \\
2\alpha & -(2\alpha + (N-2)\beta) & (N-2)\beta \\
\vdots & \ddots & \ddots \\
(N-1)\alpha & -((N-1)\alpha + \beta) & \beta \\
N\alpha & -N\alpha
\end{pmatrix} \]
Example 2: The celebrated Anick et al. model (cont’)

Q: Why is this model famous?

A: Because all quantities in

$$F(x) = \sum_{j=1}^{N} c_j e^{s_j x} v^{(j)}$$

can be obtained in explicit form!

See:

Last words

- Anick et al. model can be extended to independent non-statistical identical On-Off sources 😊

- No product-form like-result around the corner for network of fluid queues 😞