Stochastic Processes

Part 3
690PE_16 Class 4
Define n-th step transition probability matrix

We conclude

\[ P^{(n)} = [P^{(n)}_{i,j}] \]

\[ P^{(m+n)} = P^{(m)} P^{(n)} \]

These are the Chapman-Kolmogorov (CK) eqns

We have (pick \( m=1, n=k-1 \))

\[ P^{(k)} = P^{(1)} P^{(k-1)} = PP^{(k-1)} = \ldots = P^k \]

End of lecture 3
Markov chains (cont’)

In summary

\[ P(X_n = j \mid X_0 = i) \] is \((i,j)\)-entry of matrix \(P^n\)

Define \(\pi_n(j) = P(X_n = j)\) and \(\pi_n = (\pi_n(j), j \in S)\)

\[ \pi_n(j) = \sum_{i \in S} P(X_n = j \mid X_{n-1} = i) \pi_{n-1}(i), \quad n=1,2,\ldots \quad \text{(Bayes)} \]

Therefore \(\pi_n = \pi_{n-1}P, \quad n=1,2,\ldots \quad (\pi_n\text{ depends on } \pi_0)\)

**Q:** What happens as \(n \to \infty\)? Do matrix \(P^n\) and row vector \(\pi_n\) converge to something?
Markov chains (cont’)

Assume for time being that limit of $P_{i,j}^{(n)}$ exists as $n$ goes to infinity, regardless of initial value $i$

Call this limit $\pi(j) = \lim_n P(X_n = j \mid X_0 = i)$

Let $\pi = (\pi(j), j \in S)$, probability row vector

(If $S=\{0,1,2, \ldots\}$ then $\pi = (\pi(0), \pi(1), \pi(2), \ldots)$)

What should be $\pi$?

A: $\lim_n P(X_n = j \mid X_0 = i) = \lim_n P(X_n = j) = \lim_n \pi_n(j)$

$\Rightarrow \pi_n = \pi_{n-1}P$ yields $\pi = \pi P$ as $n$ goes to infinity
Markov chains (cont’)

Assume for time being that limit of $P_{i,j}^{(n)}$ exists as $n$ goes to infinity, regardless of initial value $i$

Call this limit $\pi(j) = \lim_n P(X_n = j | X_0 = i)$

Let $\pi = (\pi(j), j \in S)$, probability row vector

$\pi_n = \pi_{n-1}P$ yields $\pi = \pi P$ as $n$ goes to infinity

$\pi$ is called the stationary distribution of the MC

Also called invariant measure as if one applies matrix $P$ to it then one gets the same thing
Markov chains (cont’)

Assume for time being that limit of \( P_{i,j}^{(n)} \) exists as \( n \) goes to infinity, regardless of initial value \( i \)

**Call this limit** \( \pi(j) = \lim_n P(X_n = j \mid X_0 = i) \)

Let \( \pi = (\pi(j), j \in S) \), probability row vector

\[
\pi_n = \pi_{n-1} P \quad \text{yields} \quad \pi = \pi P \quad \text{as no goes to infinity}
\]

**Note:** We have extra (normalizing) equation

\[
\sum_{j \in S} \pi_j = 1 \quad \text{or} \quad \pi.1 = 1
\]
So, solving a MC is easy, as one only needs to solve a system of linear equations.

\[ \pi = \pi P \]
\[ \pi \cdot 1 = 1 \]

(1 = row vector with only 1’s)

Not always easy when set S is infinite or when S has many elements (complexity may be high)
Markov chains (cont’)

Q: When do we know that limit of

\[ P^{(n)}_{i,j} = P(X_n = j \mid X_0 = i) \]

exists as \( n \) goes to infinity, regardless of initial value \( i \)?
Q: Any chance that long-term behavior independent of initial state?

A: No!

- If $X_0 = 1$ then $X_n = 1$ for all $n=1,2, \ldots$
- If $X_0 = 4$ or $5$ then $X_n = 4$ or $5$
- If $X_0 = 2$ or $3$ after a while $X_n = 1$ or $X_n = 4$ or $5$
Markov chains (cont’)

A MC is irreducible if every state in its state-space is reachable from any other state

Formally: for every pair \((i,j)\), there exists \(n\) (possibly depending on \(i\) and \(j\)) so that

\[
P^{(n)}_{i,j} > 0
\]
Markov chains (cont’)

MC is periodic

\[
P(X_{3n} = 3 \mid X_0 = 3) = 1, \ n=1,2, \ldots
\]
\[
P(X_{3n+1} = 3 \mid X_0 = 3) = 0, \ P(X_{3n+2} = 3 \mid X_0 = 3) = 0, \ n=0,1,\ldots
\]

Limit of \( P_{3,j}^{(n)} \) as \( n \) goes to infinity does not exist (depends on the way infinity is approached)
Markov chains (cont’)

Define $d(i)$ as largest common divisor of all $n$ such that $P_{i,i}^{(n)} > 0$

State $i$ is aperiodic if $d(i)=1$

Ex.: $P_{i,i}^{(1)} > 0$ then $d(i)=1$. $P_{i,i}^{(2)} > 0$ and $P_{i,i}^{(3)} > 0$ $d(i)=1$

A MC is aperiodic if all states in $S$ are aperiodic ($d(i)=1$ for all $i$ in $S$)
Theorem (Invariant measure of a MC)

If a MC with transition matrix $P$ is aperiodic, irreducible, and if there exists a unique strictly positive* solution $\pi = (\pi(i), i \in S)$ to the system of linear equations

$$x = xP$$
$$x.1 = 1$$

then $\lim_n P(X_n = i)$ exists for all $i \in S$ regardless of the initial state, and $\lim_n P(X_n = i) = \pi(i)$ for all $i \in S$.

* $\pi(i) > 0$ for all $i \in S$
Markov chains: Example 1 (cont’)

\[ X_n = \# \text{ uninterrupted successes at time } n \text{ in Bernoulli trials with failure prob. } 1-p \]

Aperiodic, irreducible
Solving \( x = xP \), \( x.1 = 1 \) gives
\[ \pi_n = (1-p)p^n, \ n=0,1,... \]
= stat. prob. \( n \) consecutive successes (trivial!)

\[
P = \begin{pmatrix}
1 - p & p & 0 & 0 & 0 & \ldots \\
1 - p & 0 & p & 0 & 0 & \ldots \\
1 - p & 0 & 0 & p & 0 & \ldots \\
1 - p & 0 & 0 & 0 & p & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}
\]
Markov chains (cont’)

\[ P = \begin{pmatrix}
0.2 & 0.2 & 0.6 \\
0 & 0.5 & 0.5 \\
1 & 0 & 0
\end{pmatrix} \]

* \( P_{1,1} > 0, P_{2,2} > 0, d(1) = d(2) = 1 \)

* \( P_{3,3}^{(2)} = 0.6, P_{3,3}^{(3)} > 0, d(3) = 1 \)

\[ \Rightarrow \text{aperiodic} \]

* All states communicate

\[ \Rightarrow \text{irreducible} \]

* Solving \( x = xP \), \( x.1 = 1 \)
gives unique solution

\( x(1) = \frac{5}{11}, x(2) = \frac{2}{11}, x(3) = \frac{4}{11} \)

Solution strictly positive \( \Rightarrow \text{this is the inv. measure} \)
Simulating trajectories of this MC

After 50 time units:

\#green (state 1) = 22, \# blue (2) = 8, \# red (3) = 20
In % of time = 44% in state 1, 16% in state 2, 40% in state 3

After 100 time units:

\#green (1) = 45, \# blue (2) = 18, \# red (3) = 37
In % of time = 45% in state 1, 18% in state 2, 37% in state 3

Exact results: 45,45%, 18,18%, 36,36%
Examples

1 - Modeling of **Google PageRank algorithm**
Slides available on the class website

2 - Modeling of **IEEE 802.11 protocol**
Slides + G. Bianchi’s paper available on the class website