When preparing your written solutions, please show all steps involved in obtaining your answer. This will make it easier to assign partial credit (for partially correct answers) and will also help me (or the grader) in helping you to see where (if anywhere) you made a mistake.

1. Let $X$ be a nonnegative random variable with distribution function $F_X$ and density function $f_X$. Define the failure (hazard) rate function of $X$ to be

$$\lambda(t) = \frac{f_X(t)}{1 - F_X(t)}, \quad t \geq 0$$

We say that $X$ is an increasing in failure rate (IFR) random variable if $\lambda(t)$ is increasing (nondecreasing) in $t$ and that it is a decreasing in failure rate (DFR) random variable if $\lambda(t)$ is decreasing (non-increasing) in $t$.

(a) Show that if $X$ is an $r$-th order Erlang rv, it is an IFR rv.

(b) Show that if

$$f_X(x) = \alpha \mu_1 e^{-\mu_1 x} + (1 - \alpha) \mu_2 e^{-\mu_2 x}, \quad x \geq 0; 0 \leq \alpha \leq 1$$

then $X$ is a DFR random variable.

(c) If $X$ is an exponential rv, is it IFR or DFR?

(d) (5pt extra credit) Consider a renewal process where the interevent times are $\{X_i\}$. Let $Y$ denote the time until the next event given that we look at the process at a random point in time. If $X$ is IFR, show that $Y \leq_d X$. Similarly, show that $X \leq_d Y$ if $X$ is DFR.

2. Suppose that whether or not the NASDAQ goes up in value or not depends on what happened to the NASDAQ previously in the following manner. If it went up today and yesterday then it will go up with probability 0.7.; if it went up yesterday but not today then it will go up tomorrow with probability 0.3, if it went up today but not yesterday, it will go up tomorrow with probability 0.45, and if it went down both yesterday and today, it will go up tomorrow with probability -0.1 (just kidding ;-), I mean 0.1).

Define the state of the NASDAQ index, such that its behavior can be described by a Markov chain. Describe the meaning of each state and give the transition probability matrix for the chain.
3. Consider a discrete time Markov chain having the following transition probability matrix:

\[
P = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
.25 & .25 & .25 & .25 \\
0 & 0 & .5 & .5
\end{bmatrix}
\]

- What are the limiting probabilities \( \pi_i \) for this chain?
- What is the 2 step transition probability matrix for this chain?

4. Consider a Markov chain with transition probability matrix

\[
P = \begin{bmatrix}
p_0 & p_1 & p_2 & \cdots & p_N \\
p_N & p_0 & p_1 & \cdots & p_{N-1} \\
p_{N-1} & p_N & p_0 & \cdots & p_{N-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_1 & p_2 & p_3 & \cdots & p_0
\end{bmatrix}
\]

where \( p_0 > 0 \) and \( p_0 + p_1 + \cdots + p_N = 1 \). Determine the stationary probabilities.

5. Consider an irreducible \( L \) state discrete-state Markov chain with exponential holding times, and state space \{1, \ldots, L\}. Let \( Q = [q_{i,j}] \) be the infinitesimal generator for this MC

(a) Write, a set of equations which determine the steady state probabilities \( \pi_1, \pi_2, \ldots, \pi_L \).
(b) Given that the process is in state \( j \), what is the probability that two transitions from now it will again be in state \( j \)?
(c) What are the expected value and variance of the time the system spends during any visit to state \( j \)?
(d) Determine the average rate at which the system makes transitions into state \( j \).
(e) If we arrive at a random point in time to find the process in state \( j \), determine the density function for the time until the next transition.
(f) If we arrive at a random point in time to find the process in the steady state, determine the density function for the time until the next transition.