Problem I

Ralphie has lost his dog Ruffie in either forest A (with probability 0.4) or in forest B (with probability 0.6). If the dog is alive and not found by the N-th day of the search, it will die that evening with probability \( N/(N+2) \).

If the dog is in A (dead or alive) and Ralphie spends a day searching for it in A, the conditional probability that he will find the dog that day is 0.25. Similarly, if the dog is in B and Ralphie spends a day looking for it there, he will find the dog with probability 0.15.

The dog cannot go from one forest to the other. Ralphie can search only in the daytime, and he can travel from one forest to the other only at night.

1. In which forest should Ralphie look to maximize the probability that he finds his dog on the first day of the search?

2. Given that Ralphie looked in A on the first day but didn’t find his dog, what is the probability that the dog is in A.

3. If Ralphie flips a fair coin to determine where to look on the first day and finds the dog on the first day, what is the probability that he looked in A?

4. Ralphie has decided to look in A for the first two days. What is the probability that he will find a live dog for the first time on the second day?

5. Ralphie has decided to look in A for the first two days. Given the fact that he is unsuccessful on the first day, determine the probability that he does not find a dead dog on the second day?

6. Ralphie has found his dog on the fourth day of the search. He looked in A for the first 3 days and in B on the fourth day. What is the probability that he found his dog alive?

7. Ralphie finally found his dog late on the fourth day of the search. The only other thing we know is that he looked in A for 2 days and in B for 2 days. What is the probability he found his dog alive?

Problem II

Discrete random variable \( X \) is described by the probability mass function (PMF)

\[
p_X(x) \equiv P[X = x] = \begin{cases} 
K - x/12, & \text{if } x = 0, 1, 2 \\
0, & \text{otherwise}
\end{cases}
\]
Let $D_1, D_2, \ldots, D_N$ represent $N$ successive independent experimental values of random variable $X$.

1. Determine the numerical value of $K$.

2. Determine the probability that $D_1 > D_2$.

3. Determine the probability that $D_1 + D_2 + \cdots + D_N \leq 1.0$.

4. Define $R = \max(D_1, D_2)$ and $S = \min(D_1, D_2)$. Determine the following for all values of $s$ (and $r$ if applicable):
   
   (a) $P[S = s]$.
   (b) $P[R = s | S = 0]$.
   (c) $P[R = r, S = s]$.
   (d) $P[T = s]$ where $T = (1 + D_1)/(1 + S)$.

5. Determine the expected value and variance of random variable $S$ defined above.

6. Determine the conditional expected value and conditional variance of $S$ given $D_1 + D_2 \leq 2$.

Problem III

In this problem, we use the notation $X^*(s) = E[e^{-sX}]$ where $X$ is a real valued random variable. $X^*(s)$ is the Laplace transform of random variable $X$. Let $X$ and $Y$ be independent random variables with probability density functions

$$f_X(x) = \begin{cases} 
\lambda e^{-\lambda x}, & x \geq 0 \\
0, & x < 0 
\end{cases}$$

$$f_Y(y) = \begin{cases} 
0, & y > 0 \\
\lambda e^{\lambda y}, & y \leq 0 
\end{cases}$$

Define random variable $R$ as $R = X + Y$. Determine:

1. $X^*(s)$, $Y^*(s)$, and $R^*(s)$.

2. $E[R]$ and $\sigma^2_R$ from $R^*(s)$.

3. $f_R(r)$.

4. Repeat the previous parts for the case $R = aX + bY$. 

2