Connectivity in Cooperative Wireless Ad Hoc Networks

Abstract—**Connectivity and capacity are two measures for the performance of mobile ad hoc networks that have been studied extensively under standard point-to-point physical layer assumptions. However, extensive recent research at the physical layer has demonstrated the improvement in performance possible when multiple radios concurrently transmit in the same radio channel. In this paper, we consider how such physical layer cooperation improves the connectivity in wireless ad hoc networks. In particular, for a form of noncoherent cooperation at the physical layer, we consider conditions on the node density λ (or, equivalently, the transmit power) for full connectivity and percolation for large networks in various dimensions and with various path loss exponents α. For one-dimensional (1-D) extended networks, in sharp contrast to noncooperative networks, we demonstrate that full connectivity can be realized. In particular, for any node density with $\alpha < 1$, or for node density $\lambda > 2$ when $\alpha = 1$, full connectivity occurs with probability one. Conversely, we demonstrate that there is no percolation with probability one when $\alpha > 1$. In two-dimensional (2-D) extended networks, for any node density with $\alpha < 2$, or for node density $\lambda > 5$ when $\alpha = 2$, full connectivity is achieved. Conversely, there is no full connectivity with probability one when $\alpha > 2$, but we prove that, for $\alpha \leqslant 4$, the percolation threshold of the noncoherent cooperative network is strictly less than that of the noncooperative network. Analogous results are presented for dense networks. Hence, the main conclusion is that even relatively simple physical layer cooperation in the form of noncoherent power summing can substantially improve the connectivity of large ad hoc networks.**

I. INTRODUCTION

Wireless ad hoc networks have been a topic of extreme interest recently. Naturally, connectivity is one of the key issues that requires significant study since few network services can function properly if the network is disconnected. Although wireless ad hoc networks are finite, of course, asymptotic (in a large number of nodes) analyses have proven useful for understanding the characteristics of large networks and will be considered here. There are multiple definitions of connectivity, but two have emerged as the most often studied for large ad hoc networks. In extended networks, nodes are distributed across an infinite region according to a Poisson point process with some density λ. In dense networks, N nodes are distributed uniformly on a surface of fixed area. Connectivity for extended networks is generally defined as the existence of one cluster containing an infinite number of connected nodes, and, for dense networks, as *all* nodes being able to communicate with one another.

Conventionally, only nodes within a distance less than some r can directly communicate with each other. The transmission radius r is determined by the required decoding threshold received signal power τ, the transmission power P_t, and the path loss attenuation function. However, when a set of already-connected nodes transmits simultaneously, cooperation helps achieve the required received power, thus allowing a node to be pulled into the connected component. In Song, Goeckel and Towsley [1], noncoherent physical layer cooperation was proposed and was shown to improve the connectivity in both extended and dense networks through a combination of simulation and analysis. For extended networks, only simulation results were provided. For dense networks, analytical results were provided, but the power required for full connectivity was only modestly reduced. Here, we significantly extend the results from [1], and a nearly complete set of necessary and sufficient conditions is analytically established for extended networks with noncoherent cooperation.

For extended networks, continuum percolation with the Poisson Boolean model has been the most common approach to study connectivity. Nodes with identical range are distributed in an infinite two-dimensional space according to a Poisson point process. In particular, for a given r, when the node density λ exceeds a given threshold λ_c, there will be one infinite cluster almost surely, whereas for node densities less than λ_c there is no infinite cluster with probability one. Although the concept of percolation has been applied to many different fields, there is still no accurate analytical expression for the threshold λ_c [2]. Previous work has also shown that there is no percolation in noncooperative 1-D networks, and, through simulation, that the percolation threshold in noncooperative 2-D networks with $\pi r^2 = 1$ is approximately $\lambda_c = 4.5$ [3]. Through simulation, [1] showed that it can be lowered to some extent with the help of cooperation. Hence, there remains a need for analytical results on the existence of percolation and the value of the percolation threshold for both 1-D and 2-D networks in the cooperative case.

For dense networks, Gupta and Kumar [4] [5] set up a powerful framework for studying the connectivity of noncooperative 2-D dense networks. This framework makes use of results from continuum percolation theory to demonstrate that a power level that allows a node to connect with any other node within an area of size $(\log N + c(N))/N$ is required for full connectivity, with $c(N) \to \infty$. Specifically, when the transmission area of each single node is $\pi r^2 = \frac{\log N + c(N)}{N}$, where N is the number of nodes in the unit area disc in $\Re^2$ and $\liminf_{N\to\infty} c(N) = \infty$, the network is completely connected with probability one as $N \to \infty$. Otherwise, if $\liminf_{N\to\infty} c(N) < \infty$, there will be some isolated clusters with strictly positive probability as N approaches infinity. Therefore, the expected number of neighbors of each node has to be on the same order of $\log N$ to maintain the network connectivity. The work of [1] developed a communication model for cooperative networks, and reduced the required coverage area of each node to $4\pi(4\log N)^{\alpha/(\alpha+2)}(\log\log N + \log 2)^{2/(\alpha+2)}/N$, where $\alpha > 0$ is the path loss exponent. It breaks the necessity condition for full connectivity in [4], and conjectures that any network with path loss exponent $\alpha \leqslant 2$ for which the average number of neighbors approaches infinity is completely connected. Unfortunately, the problem of how much the power required for full connectivity can be further reduced, as well as this key latter conjecture, remains unresolved in prior studies.

Our paper focuses on such cooperative wireless ad hoc networks, and we are most concerned about the extent that cooperation can help improve the connectivity in the 1-D and 2-D cases. 1-D networks are of interest, for example, in modeling networks distributed in river valleys, whereas 2-D networks are more appropriate for open regions. For the noncoherent cooperation portion of this work, we adopt the previous cooperative framework and develop new analytical approaches that allow us to obtain percolation and full connectivity results with respect to node density λ for a given path loss exponent α in the channel model. A summary of the results is shown in Table I (c is a positive constant). Although many of the achievability results in Table 1 are for α smaller than typically encountered in wireless practice, we hasten to note that archival results such as these generally find a home; in fact, $\alpha < 2$ occurs in many 2-D communication scenarios where there is a “waveguide” effect. An excellent example of a 2-D network where this occurs is in the underwater acoustic communication channel, in which the ocean surface and ocean bottom cause such a waveguide effect. From [6], the path loss attenuation function is $r^{-\alpha}10^{-\beta(f)r}$, where $\beta(f)$ is a frequency-dependent absorption coefficient. For the practical frequency band of 8-15KHz, the absorption coefficient is small enough such that the effect of absorption is negligible for a distance up to a few kilometers. Thus, the power attenuation is mainly dictated by the power law decay with exponent $\alpha \approx 1.5$.

TABLE I
MAJOR RESULTS

Extended Networks			
1-D		2-D	
$\alpha = 1$	Percolation Full connectivity $\diamond\lambda > 2$, $r = 1$ $\diamond\lambda > 2/r$, $\forall r$	$\alpha = 2$	Percolation Full connectivity $\diamond\lambda > 5$, $r = 1$ $\diamond\lambda > 5/r$, $\forall r$
$\alpha < 1$	Percolation Full connectivity $\diamond\forall\lambda > 0$	$\alpha < 2$	Percolation Full connectivity $\diamond\forall\lambda > 0$
$\alpha > 1$	No percolation $\diamond\forall\lambda > 0$	$\alpha > 2$	No full connectivity $\diamond\forall\lambda > 0$
		$\alpha \leqslant 4$	Reduced percolation threshold
Dense Networks			
1-D		2-D	
$\alpha \leqslant 1$	Full connectivity $\diamond r = c/N$	$\alpha \leqslant 2$	Full connectivity $\diamond\pi r^2 = c/N$

The rest of this paper is organized as follows. In Section II, we first introduce some necessary background on percolation and then precisely define the cooperation model. The connectivity of cooperative wireless ad hoc networks assuming noncoherent cooperation is the core of the paper and is studied in Sections III and IV. Finally, we conclude in Section V with some comments on the problem considered and future work.

II. COOPERATION AND PERCOLATION

We extend the conventional noncooperative multi-hop framework [4], which applies to the r-radius model; that is, each node transmits the same constant power and is able to communicate directly with others (namely neighbors) within a distance of r, which is determined as follows. Let P_t be the transmission power of a single node, α be the path loss exponent, and τ be the decoding threshold (the minimum average received

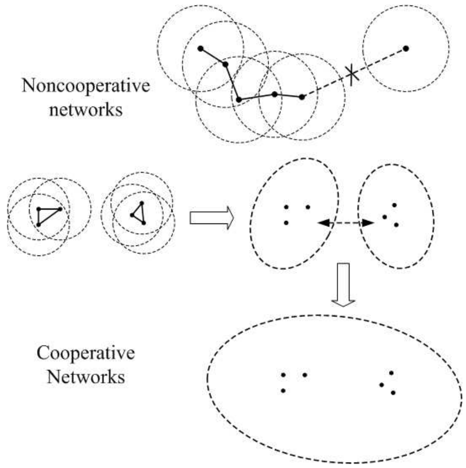

Fig. 1. Noncooperative and cooperative networks

power required to decode a transmission). For two nodes to communicate directly, they must be within distance r where r must satisfy

$$P_t \cdot r^{-\alpha} \geqslant \tau \tag{1}$$

Cooperation techniques allow clusters of nodes that have already formed under a noncooperative model to pool their resources together to further connect isolated nodes; thus, the size of each cluster keeps growing until no more nodes can be pulled into any current cluster, as shown in Figure 1.

There are a number of possible methods for realizing physical layer cooperation. For example, at the high end of performance are techniques such as distributed beamforming, which provides coherent voltage summing at the receiver by precisely phasing transmissions. Other techniques include cooperative diversity [7], distributed multiple-input multiple-output (MIMO) [8], etc. As discussed in the introduction, here we consider distributed frequency-shift keying (FSK), which employs noncoherent power summing. As will be shown, even under this set of assumptions, which requires quite achievable complexity, significant improvements in connectivity can be achieved, and these serve as lower bounds to what can be expected with other forms of cooperation.

In a noncoherent cooperative network, if a set of relay nodes, Ω, transmits simultaneously, then the average power received by node j is

$$P_t \sum_{i \in \Omega} (d_{ij})^{-\alpha}$$

where d_{ij} is the distance between node i and node j. In the worst case, all the transmission nodes in Ω are at the same distance away from the receiver. Therefore, a sufficient condition for a node to be able to connect a cluster of N connected nodes is that the distance to the furthest node in the cluster, namely $d_{N,1}$, must satisfy

$$NP_t \cdot (d_{N,1})^{-\alpha} \geqslant \tau \tag{2}$$

Thus, $(d_{N,1})^{\alpha} \leqslant Nr^{\alpha}$. Throughout this paper, we will also assume cooperation in the reception by nodes within a cluster; in particular, the aggregate power received by the cluster is used in determining whether information can be submitted successfully *to* the cluster from a remote transmitter (or cluster of transmitters). This model requires the common physical layer assumption that each receiving node is able to measure its equivalent complex gain from the transmitters, which is easily established from pilot symbols within the transmitted signal. Stated differently, receiver coherence (or the presence of receiver channel state information) is easily established, whereas transmitter coherence is challenging. The more difficult part of receiver cooperation is that, after appropriate weighting at each receiver determined by the pilot symbols, the received symbol samples must be routed to a single node in the cluster for joint processing. This certainly can be done in systems where connectivity is the critical goal, and has even been employed in systems where capacity maximization is the goal [8]. Such receiver cooperation makes all cooperative links under the noncoherent assumptions symmetric; that is, if and only if cluster X can successfully transmit to a node (or cluster of nodes) Y, that node (or cluster of nodes) Y can successfully transmit to the original cluster X. Thus, in the case of receiver cooperation, if there are two clusters of size N_1 and N_2, a sufficient condition for two clusters to be connected is that the maximum distance between any two nodes in N_1 and N_2 respectively, namely d_{N_1,N_2}, must satisfy

$$N_1 P_t \cdot (d_{N_1,N_2})^{-\alpha} \cdot N_2 \geqslant \tau \tag{3}$$

Thus, $(d_{N_1,N_2})^{\alpha} \leqslant N_1 N_2 r^{\alpha}$.

There is a bit of subtlety to the receiver cooperation that is not captured by the simple average aggregate received power model above that is used in successive sections, so here we briefly note how all of the theorems still hold when this subtlety is included. In practice, the receiver cooperation model described above provides not only power aggregation for a given link but also diversity equal to the number of nodes n_r in the receiving

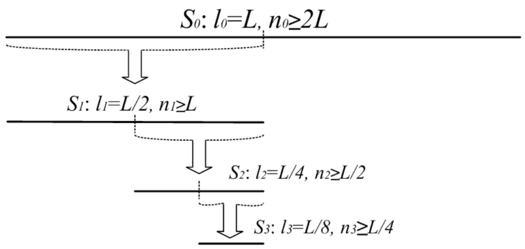

Fig. 2. An example of the dividing procedure

cluster [9, pg. 829]. Hence, links with $n_r > 1$ will perform better than the model would suggest, and thus the achievability results in Table 1 still clearly hold. In addition, the converse results rely only on establishing results for the $n_r = 1$ case, where the model is exact, so these results hold also. Hence, the results of Table I still hold for the exact physical layer model.

III. EXTENDED NETWORKS

In this section, we establish exact percolation results and connectivity laws with respect to node density λ when the path loss exponent α takes different values in 1-D and 2-D extended networks. We generally assume that each node has the same transmission radius $r = 1$, and then note how the results can be extended to the case when $r \neq 1$ in a straightforward manner.

A) One-dimensional Networks

1) Path Loss Exponent $\alpha = 1$

Lemma 1: For any segment of length L of the line, if there are $n \geqslant 2L$ nodes in this segment, cooperation guarantees that all nodes are connected.

Proof: When $L \leqslant 1$, a node can directly communicate with any other node within the segment, thus the lemma is obvious. As shown in Figure 2, when $L > 1$, divide the original segment S_0 in half, and one of the halves, call it S_1, contains $n_1 \geqslant L$ nodes.

$$n_1 \times r^\alpha \geqslant L \times r^\alpha = L = L^\alpha$$

The largest distance between any two nodes in S_0 is L. Thus, if all of the nodes in the segment S_1 are connected, which we will often term "S_1 being fully connected", it is straightforward to observe that the total transmitted power from its nodes is sufficient to allow it to connect to any node in the other half of S_0, which means that S_0 is also fully connected. Hence, we proceed to prove that S_1 is fully connected.

Now, divide S_1 in half, and similar to above, one of the halves, namely S_2, must contain $n_2 \geqslant L/2$ nodes.

$$n_2 \times r^\alpha \geqslant \frac{L}{2} \times r^\alpha = \frac{L}{2} = \left(\frac{L}{2}\right)^\alpha$$

The largest distance between any two nodes on the segment S_1 is $L/2$, and, if S_2 is fully connected, the total power of its connected nodes is sufficient for it to connect to any other node in the other half of S_1, which implies that S_1 is also connected. Hence, in turn, we proceed to prove that S_2 is fully connected.

We continue halving the interval lengths as done twice above, yielding the sequence:

$$\begin{aligned} l_k &= L/2^k \qquad\qquad k=1,2,\cdots\cdots \\ n_k &\geqslant L/2^{k-1} = 2l_k \\ n_k \times r^\alpha &\geqslant L/2^{k-1} = (l_{k-1})^\alpha \end{aligned}$$

where l_k is the length of S_k. Repeating the argument employed above, S_k is connected as long as S_{k+1} is connected.

For any fixed finite L, there must exist some k_c ($k_c \geqslant 0$) such that

$$\begin{aligned} l_{k_c} &= L/2^{k_c} > 1 \\ l_{k_c+1} &= L/2^{k_c+1} \leqslant 1 \\ n_{k_c+1} &\geqslant 2l_{k_c+1} = L/2^{k_c} > 1 \end{aligned}$$

Since $l_{k_c+1} \leqslant 1$ and the number of nodes within it satisfies $n_{k_c+1} \geqslant 2$, the nodes within S_{k_c+1} are clearly connected. Therefore, S_{k_c} is completely connected, which implies that S_{k_c-1} is completely connected, etc. Applying this argument k_c times, we conclude that S_0, the original segment of length L, is completely connected. ■

Remark 1: Generally, the transmission radius r does not necessarily equal 1, but this is easily addressed through an analogous argument. In this case, for any fixed finite r and L, if there are $n \geqslant 2L$ nodes in a segment of length rL, cooperation guarantees that all nodes are connected.

Lemma 1 provides an explicit construction for how nodes can cooperate to realize connectivity. This will be the basis not only for the following key theorem, but also for its analogous version in the 2-D case.

Theorem 1: In a 1-D extended network with $\alpha = 1$ and transmission radius $r = 1$, if the node density $\lambda > 2$, percolation and full connectivity occur with probability one.

Proof: A one-dimensional network can be written as the union of an infinite number of adjacent segments of length L. According to Lemma 1, for any instantiation for which there are at least $2L$ nodes in any segment of length L, cooperation guarantees that all nodes within it are connected.

Let n_k be the number of nodes in segment k, N be the total number of segments, and node density $\lambda = 2 + \varepsilon$, where $\varepsilon > 0$. The number of nodes in a segment n_k is a Poisson random variable of parameter $\mu = \lambda L$, thus for any $\delta \in (0, 1]$, Chernoff's bound yields

$$P\left(n_k < (1-\delta)\mu\right) < \exp\left(-\frac{\mu\delta^2}{2}\right) \qquad (4)$$

where $\mu = E[n_k] = \lambda L = (2+\varepsilon)L$. Let $\delta = \varepsilon/(2+\varepsilon)$, and we have

$$P\left(n_k < 2L\right) < \exp\left(-\frac{\varepsilon^2}{2(2+\varepsilon)}L\right) = \exp(-\beta L)$$

where $\beta = \varepsilon^2/[2(2+\varepsilon)]$. Each segment is an independent interval from an identical Poisson point process; therefore,

$$\begin{aligned} P\left(n_k \geqslant 2L, \text{ all } k\right) &= \left(1 - P\left(n_k < 2L\right)\right)^N \\ &\geqslant \left(1 - \exp(-\beta L)\right)^N \end{aligned}$$

Let $L = 2\log N/\beta$ be a function of N, thus $\exp(-\beta L) = 1/N^2$, and

$$P(n_k \geqslant 2L, \text{ all } k) \geqslant \left(1 - \frac{1}{N^2}\right)^N \xrightarrow{N\to\infty} 1 \qquad (5)$$

As the number of segments tends to infinity, they cover the whole line, and with probability one, every one of the segments is completely connected (i.e. for any segment, all of the nodes within that segment are connected).

Finally, when a segment of length L contains at least $2L$ nodes, it is able to connect adjacent segments.

$$(2L) \cdot r^\alpha = 2L = (2L)^\alpha$$

Thus, all of the nodes on the line are connected, which means both percolation and full connectivity are achieved. ■

Remark 2: Generally, when the transmission radius $r \neq 1$, if the node density $\lambda > 2/r$, we have the same result as above.

2) Path Loss Exponent $\alpha < 1$

Theorem 2: In a 1-D extended network with $\alpha < 1$ and any finite node density $\lambda > 0$, percolation and full connectivity occur with probability one.

Proof: First, consider $2^N - 1$ segments of length L in a 1-D network, where N is a positive integer. For any

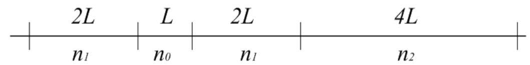

Fig. 3. Dividing a 1-D network

finite node density $\lambda > 0$, define $\lambda = \theta + \varepsilon$, where ε is an arbitrarily small positive number; thus, $\lambda > \theta > 0$. Let $L = \log\log N/\gamma$, and C represent the event that such a segment is fully connected, where $\gamma = \varepsilon^2/[2(\theta+\varepsilon)]$.

For any finite $\theta > 0$, we can always find some N that is large enough to satisfy $\theta \geqslant 1/L^{1-\alpha}$. If there are θL nodes within a unit length of the segment, they are not only completely connected themselves, but also able to connect all of the other nodes within L.

$$(\theta L) \cdot r^\alpha = \theta L \geqslant L^\alpha$$

The nodes are uniformly distributed within each segment, thus,

$$P(C) \geqslant (1/L)^{\theta L}$$

The probability that there exists at least one completely connected segment of length L is

$$\begin{aligned} & P(\exists \text{ fully connected segment of length } L) \\ & = 1 - \left(1 - P(C)\right)^{2^N - 1} \\ & \geqslant 1 - \left(1 - (1/L)^{\theta L}\right)^{2^N - 1} \\ & = 1 - \left(1 - \left(\frac{\gamma}{\log\log N}\right)^{(\theta \log\log N/\gamma)}\right)^{2^N - 1} \\ & \xrightarrow{N\to\infty} 1 \end{aligned} \qquad (6)$$

Thus, we can find a fully connected segment of length L on the line with probability one.

Suppose we start from such a segment and divide the network into an infinite number of adjacent segments of exponentially growing length in both directions, so that a listing of the segment lengths would be $\cdots 8L, 4L, 2L, L, 2L, 4L, 8L \cdots$, where the L in the center corresponds to the fully connected segment found above. We assign a number to a given segment by the number of segments between it and the starting segment of length L; hence, corresponding to the segment lengths above, the numbers would be $\cdots 3, 2, 1, 0, 1, 2, 3 \cdots$. Let n_k be the number of nodes in segment k, with k representing the sequence number, and l_k the segment length, $l_k = 2^k L, k = 0, 1, 2, \ldots, (N-1)$, as shown in Figure 3. N is the total number of such segments.

Let $\delta = \varepsilon/(\theta+\varepsilon) \in (0, 1]$. Similarly, apply Chernoff's bound with $\mu = E[n_k] = \lambda 2^k L = (\theta+\varepsilon)2^k L$ and we have,

$$P\left(n_k < 2^k\theta L\right) < \exp\left(-\frac{\varepsilon^2}{2(\theta+\varepsilon)}2^k L\right) = \exp\left(-\gamma 2^k L\right)$$

where $\gamma = \varepsilon^2/[2(\theta+\varepsilon)]$. Also, each segment is an independent interval of an identical Poisson point process; thus,

$$P\left(n_k \geqslant 2^k\theta L, \text{ all } k\right) \geqslant \prod_{k=0}^{N-1}\left(1-\exp(-\gamma 2^k L)\right)$$

Since $L = \log\log N/\gamma$, $\exp(-\gamma 2^k L) = 1/(\log N)^{2^k}$, thus for all k, when $N \to \infty$,

$$P\left(n_k \geqslant 2^k\theta L, \text{ all } k\right) \geqslant \prod_{k=0}^{N-1}\left(1-\frac{1}{(\log N)^{2^k}}\right) \xrightarrow{N\to\infty} 1$$

The above result gives a lower bound on the node count in every segment.

As N tends to infinity and the union of the segments covers the whole line, the event that, for all k, the k^{th} segment contains at least $2^k\theta L$ nodes occurs with probability one. In order to connect the adjacent segments, n_k must satisfy

$$\begin{aligned} n_k &\geqslant 2^k\theta L \geqslant \left(2^k+2^{k+1}\right)^\alpha L^\alpha \\ \theta &\geqslant \frac{3^\alpha}{\left(2^k L\right)^{1-\alpha}} = \frac{3^\alpha\gamma^{1-\alpha}}{\left(2^k\log\log N\right)^{1-\alpha}} \xrightarrow{N\to+\infty} 0 \end{aligned}$$

For any finite node density $\lambda > 0$, there exists one fully connected segment of length L, and then, at each step, a segment in the sequence is able to connect the next segment in the sequence. Thus, all of the nodes are connected, and percolation and full connectivity occur. ∎

Remark 3: Generally, the above result also holds with transmission radius $r \neq 1$. In particular, it is straightforward to show that that for $\alpha < 1$, any $r > 0$ and any finite node density $\lambda > 0$, percolation and full connectivity occur simultaneously with probability one.

3) Path Loss Exponent $\alpha > 1$

Theorem 3: In a 1-D extended network with $\alpha > 1$ and any node density $\lambda > 0$, percolation never occurs.

Proof: Consider a 1-D network where nodes are distributed according to a Poisson point process of constant rate λ. Pick a node x on the line, and let $\{x_j\}_{j=-\infty}^{\infty}$ represent the positions of all the other nodes. Assume all nodes except x are connected and ρ is the maximum possible power x receives. Thus,

$$\rho(x) = P_t\sum_{j=-\infty}^{\infty}\frac{1}{\left(d_{x,x_j}\right)^\alpha}$$

This is an upper bound of transmission power for all cooperating clusters. When $\alpha > 1$, $\rho(x)$ converges and its probability density function $f_\rho(y)$ is that of a Levy-stable random variable [12]. Thus,

$$P\left(\rho(x) < \tau\right) = \int_0^\tau f_\rho(y)\,dy = P_0 > 0 \tag{7}$$

Therefore, node x cannot be reached with non-zero probability even if all other nodes cooperate.

Consider a segment of length L on the line. Given the node density $\lambda > 0$, let $\lambda = \lambda_0 + \varepsilon$, where $\varepsilon > 0$ is an arbitrarily small positive value. Let n and n_0 represent the number of total and isolated nodes in segment L respectively. Pick $\delta = (\lambda-\lambda_0)/\lambda \in (0,1]$, apply Chernoff's bound with $\mu = \lambda L$,

$$P(n \geqslant \lambda_0 L) \geqslant 1 - \exp\left(-\frac{(\lambda-\lambda_0)^2}{2\lambda}L\right) \xrightarrow{L\to+\infty} 1$$

Define a Bernoulli random variable X_i for the connectivity state of each node in segment L, where $i = 1, 2, \cdots, N$,

$$X_i = \begin{cases} 1, & \text{isolated;} \\ 0, & \text{connected.} \end{cases}$$

Then the expected number of isolated nodes is

$$\begin{aligned} E[n_0] = E\left[\sum_{i=1}^n X_i\right] &= \sum_{i=1}^n P(X_i = 1) \\ &\geqslant \sum_{i=1}^n P_0 = nP_0 \geqslant \lambda_0 L P_0 \end{aligned}$$

Let $\delta = 1/2$ and apply Chernoff's bound with $\mu = E[n_0]$,

$$P\left(n_0 \geqslant \frac{1}{2}nP_0\right) \geqslant 1 - \exp\left(-\frac{\lambda_0 L P_0}{8}\right) \xrightarrow{L\to+\infty} 1 \tag{8}$$

Divide the 1-D network into an infinite number of adjacent segments of length L. Let N be the total number of such segments, and let $L = \log N$. As $N \to \infty$ and the segments cover the whole line, L also goes to infinity, and there are at least $\lambda_0 L P_0/2 \to \infty$ isolated nodes with probability one in each segment. Therefore, for $\alpha > 1$, any $r > 0$ and any $\lambda > 0$, percolation (and, hence, full connectivity) never occurs. ∎

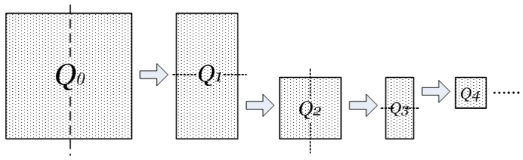

Fig. 4. An example of the dividing procedure

B) Two-dimensional Networks

1) Path Loss Exponent $\alpha = 2$

Lemma 2: For any area of size $L \times L$, if there are $n \geqslant 5L^2$ nodes in this area, cooperation guarantees that all the nodes are completely connected.

Proof: Per above, assume each node's transmission radius is $r = 1$. When $L \leqslant 1/\sqrt{2}$, a node can directly communicate with any other node within the square, thus the lemma is obvious. When $L > 1/\sqrt{2}$, divide the original square, Q_0, in the following manner, as shown in Figure 4:

(1) Divide Q_0 into two equal rectangles, one of which, namely Q_1, contains $n_1 \geqslant 5L^2/2 > 2L^2$ nodes.

$$n_1 \times r^\alpha > 2L^2 \times r^\alpha = 2L^2 = \left(\sqrt{2}L\right)^\alpha$$

The largest distance between any two nodes in Q_0 is $\sqrt{2}L$. Thus, if all of the nodes in the square Q_1 are connected, which we will often term "Q_1 being fully connected", it is straightforward to observe that the total transmitted power from its nodes is sufficient to allow it to connect to any node in the other half of Q_0, which means that Q_0 is also fully connected. Hence, we proceed to prove that Q_1 is fully connected.

(2) Continue to divide Q_1 into two equal squares, one of which, namely Q_2, contains $n_2 \geqslant 5L^2/4$ nodes.

$$n_2 \times r^\alpha \geqslant \frac{5L^2}{4} \times r^\alpha = \frac{5L^2}{4} = \left(\frac{\sqrt{5}L}{2}\right)^\alpha$$

The largest distance between any two nodes in Q_1 is $\sqrt{5}L/2$, and, if Q_2 is fully connected, the total power of its connected nodes is sufficient for it to connect to any other node in the other half of Q_1, which implies that Q_1 is also fully connected. Hence, in turn, we proceed to prove that Q_2 is fully connected.

(3) We continue dividing the squares/rectangles as done twice above, yielding the sequence:

$$\begin{aligned} a_k &= L^2/2^k \qquad k\text{=1,2,}\cdots\cdots \\ n_k &\geqslant 5L^2/2^k \end{aligned}$$

where a_k is the area of Q_k. Define d_k to be the diagonal length of Q_k, and we have

$$d_k = \begin{cases} \sqrt{2}L/2^{k/2}, & k \text{ is even;} \\ \sqrt{5}L/2^{(k+1)/2}, & k \text{ is odd.} \end{cases}$$

Therefore,

$$n_k \times r^\alpha \begin{cases} \geqslant \left(\sqrt{5}L/2^{k/2}\right)^2 = (d_{k-1})^\alpha, & k \text{ is even;} \\ > \left(\sqrt{2}L/2^{(k-1)/2}\right)^2 = (d_{k-1})^\alpha, & k \text{ is odd.} \end{cases}$$

Obviously, the nodes in Q_k are connected as long as the nodes in Q_{k+1} are all connected. For any fixed finite L, there must exist some k_c ($k_c \geqslant 0$) such that

$$\begin{aligned} d_{k_c} &> 1 \\ d_{k_c+1} &\leqslant 1 \\ n_{k_c+1} &\geqslant (d_{k_c})^2 > 1 \end{aligned}$$

Since $d_{k_c+1} \leqslant 1$ and the number of nodes in Q_{k_c+1} satisfies $n_{k_c+1} \geqslant 2$, it is clearly all connected. Therefore, we conclude that all the nodes in Q_0, the original square of size $L \times L$, are connected. ∎

Theorem 4: In a 2-D extended network with $\alpha = 2$ and transmission radius $r = 1$, both percolation and full connectivity occur with probability one when the node density $\lambda > 5$.

Proof: Similar to the 1-D case, a 2-D network can be divided into an infinite number of adjacent squares of size $L \times L$. According to Lemma 2, if there are at least $5L^2$ nodes in any square, cooperation guarantees the nodes within it are all connected. The proof then follows that of Theorem 2 to establish that there exists a sequence of square sizes $L \times L$ such that both the squares asymptotically cover the entire plane, and, with probability one, every square contains at least $5L^2$ nodes. It is also straightforward to observe that when a square of size $L \times L$ contains at least $5L^2$ nodes, it is able to connect all eight adjacent squares. Thus all the nodes on the plane are connected, which means both percolation and full connectivity are achieved. The detailed proof can be found in Wang et al. [10]. ∎

2) Path Loss Exponent $\alpha < 2$

Theorem 5: In a 2-D extended network with $\alpha < 2$ and finite node density $\lambda > 0$, both percolation and full connectivity occur with probability one.

Proof: First, consider 4^{N-1} squares of size $L \times L$ in a 2-D network, where N is a positive integer. For any

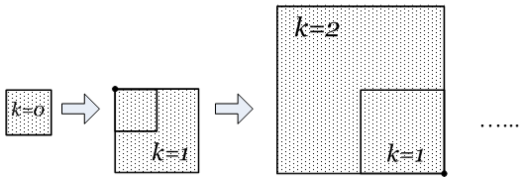

Fig. 5. Dividing a 2-D network

finite node density $\lambda > 0$, define $\lambda = \theta + \varepsilon$, where ε is an arbitrarily small positive number. Let C represent the event that such a square is fully connected, and $L = \sqrt{\log\log N/\gamma}$, where $\gamma = \varepsilon^2/[2(\theta+\varepsilon)]$. The nodes are uniformly distributed within each square, thus,

$$P(C) \geqslant \left(1/2L^2\right)^{\theta L^2}$$

The probability that there exists at least one completely connected square of size $L \times L$ is

$$\begin{aligned} & P(\exists \text{ fully connected square of size } L \times L) \\ & \geqslant 1 - \left(1 - \left(\frac{\gamma}{2\log\log N}\right)^{(\theta \log\log N/\gamma)}\right)^{4^{N-1}} \\ & \xrightarrow{N\to\infty} 1 \end{aligned} \tag{9}$$

Thus, we can find a fully connected square of size $L \times L$ with probability one.

Suppose we start from such a square and divide the network into an infinite number of overlapping squares of exponentially growing size, so that a listing of the square areas would be $L^2, 4L^2, 16L^2 \cdots$, where the $L \times L$ corresponds to the fully connected square found above. From this starting square, we pick its upper left corner and draw a square to the lower right of size $2L \times 2L$. Then, we jump to the lower right corner of the second square and draw a square to the upper left of size $4L \times 4L$. Then, we jump to the upper left corner of the third square and draw a square to the lower right of size $8L \times 8L$, etc., as shown in Figure 5. The proof then proceeds analogously to that of Theorem 2, and can be found in Wang et al. [10]. ■

3) Path Loss Exponent $\alpha > 2$

Both percolation and full connectivity are important in extended networks. First, we demonstrate that, for $\alpha > 2$, full connectivity does not occur with probability one. However, even for the noncooperative model, it has been proven that there exists a percolation threshold. An infinite cluster appears almost surely if the node density exceeds this threshold, and there is no infinite cluster almost surely if the node density is below this threshold [1] [2] [11]. Here, we demonstrate that noncoherent cooperation results in a percolation threshold strictly less than that for the noncooperative case.

Theorem 6: In a 2-D extended network with $\alpha > 2$ and any node density $\lambda > 0$, full connectivity never occurs.

Proof: Pick a node (x, y) on the plane, and let $\{(x_j, y_j)\}_{j=-\infty}^{\infty}$ represent the node position of all other nodes. Assume the other nodes are all connected, in which case the maximum possible power (x, y) receives is

$$\rho(x,y) = P_t \sum_{j=-\infty}^{\infty} \frac{1}{\left(d_{(x,y),(x_j,y_j)}\right)^{\alpha}}$$

This is an upper bound of transmission power for all cooperating clusters. When $\alpha > 2$, $\rho(x, y)$ converges and its probability density function $f_\rho(z)$ is also that of a Levy-stable random variable [12]. Thus,

$$P\left(\rho(x,y) < \tau\right) = \int_0^{\tau} f_\rho(z)\, dz = P_1 > 0 \tag{10}$$

Therefore, for all cooperating clusters of any size, there must exist some point that cannot be reached with non-zero probability.

The proof then follows analogously to Theorem 3, with an $L \times L$ square in the plane replacing the segment of length L in the line. More details can be found in Wang et al. [10]. ■

Note that there still exists percolation in some cases although full connectivity never occurs. For the Poisson Boolean model, denote by λ_c the critical intensity for percolation. We consider $r = 1$, and normalize the power so that two nodes connect if the received power is greater than $1/r^\alpha = 1$.

Denote $f(u, \alpha)$ to be the function

$$f(u,\alpha) \quad \triangleq \quad \frac{1}{u^\alpha} + \frac{1}{(1+u)^\alpha} \tag{11}$$

and $g(\alpha)$ to be the value of u such that $f(g(\alpha), \alpha) = 1$. Since $f(\cdot, \alpha)$ is decreasing, $f(1, \alpha) > 1$ and $\lim_{u\to\infty} f(u, \alpha) = 0$ for all α, $g(\alpha)$ is well defined. Actually, $g(\alpha) > 1$, and it can be observed that $g(\alpha)$ is decreasing as a function of α.

Theorem 7: In a 2-D extended network with $0 < \alpha \leqslant 4$, cooperation reduces the critical intensity by a factor $1 - \epsilon(\alpha)$, with $\epsilon(\alpha) > 0$.

Proof: We conjecture that the Theorem is actually true for all α. It is sufficient to consider here a restricted form of cooperation, where only *pairs* of node cooperate if they are within distance 1 of each other.

Consider a point x of the underlying Poisson process. If x has a neighbor y within distance 1, then x and y can jointly connect with nodes further away. The power received by a node z from the pair (x, y) is

$$\begin{aligned} \frac{1}{d_{zx}^{\alpha}} + \frac{1}{d_{zy}^{\alpha}} &\geqslant \frac{1}{d_{zx}^{\alpha}} + \frac{1}{(d_{zx} + d_{xy})^{\alpha}} \\ &\geqslant f(d_{zx}, \alpha) \end{aligned} \tag{12}$$

by the triangular inequality, and since $d_{xy} \leqslant 1$. Thus, if $d_{zx} \leqslant g(\alpha)$, then the power received at z is greater than 1, and the pair (x, y) can cooperate to connect with z.

A point x can hence connect to any point in the ring centered at x of radius between 1 and $g(\alpha)$ if it has a neighbor in the circle of radius one; that is, with probability that there exists a point in the Poisson process within distance 1, namely $1 - \exp(-\lambda\pi)$. It can also connect to a point in the circle of center x and radius one with the same probability. Thus, with probability $1 - \exp(-\lambda\pi)$, x can connect to a node within distance $g(\alpha)$.

This means that we can couple our pair-wise cooperative model to a Poisson boolean model by removing nodes with probability $1 - \exp(-\lambda\pi)$ from the underlying Poisson process, and replacing the connection area at each remaining node by a disk of radius $g(\alpha)$. Note that this new Poisson boolean model will have a lesser connectivity than the cooperative model, due to the inequalities in (12), and thus, if this Poisson boolean model percolates for a given intensity λ, so does the cooperative model. We now need to show the Poisson boolean model percolates for $\lambda < \lambda_c$.

We have constructed a Poisson boolean model with intensity $\lambda(1 - \exp(-\lambda\pi))$ and fixed connectivity radius $g(\alpha)$. This is equivalent to a Poisson boolean model with radius 1 and intensity $\lambda(1 - \exp(\lambda\pi))g(\alpha)^2$, as for any $\gamma > 0$, a Poisson boolean model (λ, r) is equivalent to another one with parameters $(\gamma^2\lambda, r/\gamma)$. Define $h(\lambda, \alpha) \triangleq \lambda(1 - \exp(-\lambda\pi))g(\alpha)^2$. Our constructed Poisson boolean model percolates if $h(\lambda, \alpha) > \lambda_c$.

Consider now $\alpha = 4$. We need to show that for λ close to λ_c, $(1 - \exp(-\lambda\pi))\, g(\alpha)^2 > 1$. If this is true, then by continuity of the function h, we can choose $\epsilon(\alpha)$ such that $h(\lambda_c(1 - \epsilon(\alpha)), \alpha) > \lambda_c$. Substituting in the value $\lambda_c\pi = 4.5$ and $g(4) = 1.0157$ gives:

$$(1 - \exp(-\lambda\pi))g(\alpha)^2 = 1.02 \tag{13}$$

The value of λ_c is approximate, but $(1 - \exp(-\lambda\pi))g(\alpha)^2$ is above 1.018 for all $\lambda\pi$ taking value in $(4.4, 4.6)$, which does include λ_c. This proves the theorem for $\alpha = 4$, taking $\epsilon(4) = 0.01$. Since $g(\alpha)$ is decreasing in α, the theorem is also true for all lesser values of $\alpha \leqslant 4$. For $\alpha = 3$, $\epsilon(3) = 0.06$ can be chosen, and for $\alpha = 2$, $\epsilon(2) = 0.19$. ■

IV. DENSE NETWORKS

Here, the goal is to find the smallest transmission power (as a function of N) such that complete connectivity is maintained as $N \to \infty$ for N nodes uniformly distributed in a unit area. It is apparent that the majority of the results for extended networks have analogs for dense networks, although the technical details are sometimes more complicated due to edge effects.

Theorem 8: In a 1-D dense network with $\alpha \leqslant 1$, there exists a sequence of transmission ranges r of order $O(1/N)$, where $N \to \infty$ is the total number of nodes within the unit segment, such that full connectivity always occurs.

Proof: We consider a segment of unit length with N nodes and perform the same division procedure as shown in Figure 2. Assuming a transmission radius $r = 2/N$, we have

$$\begin{aligned} l_k &= 1/2^k \qquad\qquad k{=}1,2,\cdots\cdots \\ n_k &\geqslant N/2^k = N l_k \end{aligned}$$

where l_k and n_k are respectively the length and the node count of the segment S_k. After a finite number of steps of dividing, there exists a k_c ($k_c \geqslant 0$) such that

$$\begin{aligned} l_{k_c} &= 1/2^{k_c} > r \\ l_{k_c+1} &= 1/2^{k_c+1} \leqslant r \\ n_{k_c+1} &\geqslant N l_{k_c+1} > 1 \end{aligned}$$

Since $l_{k_c+1} \leqslant r$ and $n_{k_c+1} > 1$, the nodes within S_{k_c+1} are clearly connected. For $k = 1, 2, \cdots, k_c + 1$,

$$n_k \cdot r^{\alpha} \geqslant N l_k (2/N)^{\alpha} = N^{1-\alpha} 2^{\alpha - k}$$

Since $\alpha \leqslant 1$, $N > 2^{k_c+1}$ and $k \leqslant k_c + 1$,

$$N^{1-\alpha} 2^{\alpha-k} \cdot 2^{(k-1)\alpha} \geqslant 2^{(1-\alpha)(k_c+1-k)} \geqslant 1$$

Thus, $n_k \cdot r^{\alpha} \geqslant (l_{k-1})^{\alpha}$, and all of the nodes within S_0, the original segment of unit length, are completely connected with transmission range $r = 2/N$. ■

Theorem 9: In a 2-D dense network with $\alpha \leqslant 2$, there exists a sequence of transmission areas of order $O(1/N)$, where $N \to \infty$ is the total number of nodes within the unit area, such that full connectivity always occurs.

Proof: We consider a square of unit area with N nodes and perform the same division procedure shown in Figure 4. Assume $r^2 = 5/N$, and we have

$$\begin{aligned} a_k &= 1/2^k \qquad\qquad k{=}1,2,\cdots\cdots \\ n_k &\geqslant N/2^k = Na_k \end{aligned}$$

a_k is the area, d_k is the diagonal and n_k is the corresponding node count. Similarly, there exists some k_c ($k_c \geqslant 0$) such that

$$\begin{aligned} d_{k_c} &> r \\ d_{k_c+1} &\leqslant r \end{aligned}$$

$$n_{k_c+1} \geqslant Na_{k_c+1} \begin{cases} > \left(r^2/4\right) N > 1, & k_c \text{ is even;} \\ > \left(r^2/5\right) N = 1, & k_c \text{ is odd.} \end{cases}$$

Since $d_{k_c+1} \leqslant r$ and $n_{k_c+1} > 1$, the nodes within it are clearly connected. For $k = 1, 2, \cdots, k_c + 1$,

$$n_k \cdot r^\alpha \geqslant Na_k(5/N)^{\alpha/2} = 2^{-k}5^{\frac{\alpha}{2}}N^{1-\frac{\alpha}{2}}$$

Since $\alpha \leqslant 2$, $Na_{k_c+1} > 1$ and $k \leqslant k_c + 1$,

$$\begin{cases} 2^{-k}5^{\alpha/2}N^{1-\alpha/2} \cdot \left(2^k/5\right)^{\frac{\alpha}{2}} > 1, & k \text{ is even;} \\ 2^{-k}5^{\alpha/2}N^{1-\alpha/2} \cdot \left(2^{k-2}\right)^{\frac{\alpha}{2}} > 1, & k \text{ is odd.} \end{cases}$$

Thus, $n_k \cdot r^\alpha \geqslant (d_{k-1})^\alpha$, and all of the nodes within Q_0, the original square of unit area, are completely connected with transmission area $\pi r^2 = 5\pi/N$. ■

V. CONCLUSION

In this paper, we have shown that physical layer cooperation is able to significantly improve the connectivity in wireless ad hoc networks. Consider large ad hoc wireless networks with path loss exponent α, transmission range r, and node density λ under a noncoherent cooperation model. For 1-D extended networks, percolation and full connectivity can be achieved with probability one in the case that $\alpha = 1$, $\lambda > 2/r$, $\forall r$ or $\alpha < 1$, $\forall\lambda$, $\forall r$. There is no percolation with probability one in the case when $\alpha > 1$. Similarly, for 2-D extended networks, percolation and full connectivity can be achieved with probability one in the case when $\alpha = 2$, $\lambda > 5/r$, $\forall r$ or $\alpha < 2$, $\forall\lambda$, $\forall r$. There is no full connectivity with probability one in the case when $\alpha > 2$, but we have shown that cooperation reduces the threshold above which percolation occurs. For dense networks, it is straightforward to apply these results to establish the conjecture from [1] that $O(1/N)$ transmission area is sufficient for complete connectivity with probability one when $\alpha \leqslant 2$ in the 2-D case.

In this paper we have assumed a path loss attenuation function in the form of a power law, i.e., $l(r) = r^{-\alpha}$, where r is the distance between the sender and receiver, and α is the path loss exponent. Note that this attenuation function has a singularity at the origin, and the received power increases without bound as r decreases. However, in reality, the received power is always finite and bounded from above by the transmission power. To reflect the finite received power, a more realistic path loss attenuation function is $l(r) = \min(1, r^{-\alpha})$ [13]. It can be easily verified that the proofs in the paper do not rely on the increasing scaling property of the power law function as r decreases. On the contrary, we consider the worst case scenario where the tail behavior of the power law for large r is used. Thus, we have verified that the results still hold under the more realistic bounded path loss attenuation function.

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