

Introduction to *Game Theory*

Game Theory Seminar Lecture 1

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What is Game Theory About?

- ❑ Analysis of situations where conflict of interests are present



- ❑ Game of Chicken
 - driver who steers away loses
- ❑ What should drivers do?
- ❑ Goal is to prescribe how conflicts can be resolved

Applications of Game Theory

- Theory developed mainly by mathematicians and economists
 - contributions from biologists
- Widely applied in many disciplines
 - from economics to philosophy, including computer science (AI)
 - goal is often to understand some phenomena
- "Recently" applied to computer networks
 - Nagle, RFC 970, 1985
 - "datagram networks as a multi-player game"
 - wider interest starting around 2000

Limitations of Game Theory

- ❑ No unified solution to general conflict resolution
- ❑ Real-world conflicts are complex
 - models can at best capture important aspects
- ❑ Players are considered rational
 - determine what is best for them given that others are doing the same
- ❑ No unique prescription
 - not clear what players should do
- ❑ But it can provide intuitions, suggestions and partial prescriptions
 - the best mathematical tool we have

What is a Game?

- A Game consists of
 - at least two players
 - a set of strategies for each player
 - a preference relation over possible outcomes
- Player is general entity
 - individual, company, nation, protocol, animal, etc
- Strategies
 - actions which a player chooses to follow
- Outcome
 - determined by mutual choice of strategies
- Preference relation
 - modeled as utility (payoff) over set of outcomes

Classification of Games

- ❑ Many types of games
 - three major categories
- ❑ Non-Cooperative (Competitive) Games
 - individualized play, no bindings among players
- ❑ Cooperative Games
 - play as a group, possible bindings
- ❑ Repeated and Evolutionary Games
 - dynamic scenario

Matrix Game (Normal form)

Strategy set for Player 1

Player 1

Player 2

Strategy set for Player 2

		Player 2		
		A	B	C
Player 1	A	(2, 2)	(0, 0)	(-2, -1)
	B	(-5, 1)	(3, 4)	(3, -1)

Payoff to Player 1

Payoff to Player 2

□ Simultaneous play

- players analyze the game and then write their strategy on a piece of paper

More Formal Game Definition

□ Normal form (strategic) game

- a finite set N of players
- a set strategies A_i for each player $i \in N$
- payoff function $u_i(s)$ for each player $i \in N$
 - where $s \in A = \times_{j \in N} A_j$ is the set of strategies chosen by all players

□ A is the set of all possible outcomes

□ $s \in A$ is a set of strategies chosen by players

- defines an outcome

□ $u_i : A \rightarrow \mathcal{R}$

Two-person Zero-sum Games

- One of the first games studied
 - most well understood type of game
- Players interest are strictly opposed
 - what one player gains the other loses
 - game matrix has single entry (gain to player 1)
- Intuitive solution concept
 - players maximize gains
 - unique solution

Analyzing the Game

- Player 1 maximizes matrix entry, while player 2 minimizes

		Player 2			
		A	B	C	D
Player 1	A	12	-1	1	0
	B	3	1	4	-18
	C	5	2	4	3
	D	-16	0	1	-1

Strictly dominated strategy (dominated by C)

Weakly dominated strategy (dominated by B)

Dominance

- Strategy S *strictly dominates* a strategy T if every possible outcome when S is chosen is better than the corresponding outcome when T is chosen.

- Dominance Principle
 - rational players never choose dominated strategies

- Removal of strictly dominated strategies
 - iterated removal

Analyzing the Reduced Game

		Player 2		
		A	B	D
Player 1	A	12	-1	0
	C	5	2	3
	D	-16	0	-1

Outcome (C, B)
seems "stable"

- saddle point of game

Saddle Points

- An outcome is a *saddle point* if the outcome is both less than or equal to any value in its row and greater than or equal to any value in its column

- Saddle Point Principle
 - Players should choose outcomes that are saddle points of the game

- Value of the game
 - value of saddle point entry if it exists

Why Play Saddle Points?

- If player 1 believes player 2 will play X
 - player 1 should play best response to X
- If player 2 believes player 1 will play Y
 - player 2 should play best response to Y

- Why should player 1 believe player 2 will play X?
 - playing X guarantees player 2 loses at most v
- Why should player 2 believe player 1 will play Y?
 - playing Y guarantees player 1 wins at least v

Powerful arguments to play saddle point

Solving the Game (min-max algorithm)

		Player 2				
		A	B	C	D	
Player 1	A	4	3	2	5	2
	B	-10	2	0	-1	-10
	C	7	5	1	3	1
	D	0	8	-4	-5	-5
		7	8	2	5	

- choose maximum entry in each column
- choose the minimum among these
- this is the minimax value
- choose minimum entry in each row
- choose the maximum among these
- this is maximin value
- if $\text{minimax} = \text{maximin}$, then this is the saddle point of game

Multiple Saddle Points

- In general, game can have multiple saddle points

		Player 2				
		A	B	C	D	
Player 1	A	3	2	2	5	2
	B	2	-10	0	-1	-10
	C	5	2	2	3	2
	D	8	0	-4	-5	-5
		8	2	2	5	

- Same payoff in *every* saddle point
 - unique value of the game
- Strategies are interchangeable
 - Example: strategies (A, B) and (C, C) are saddle points then (A, C) and (C, B) are also saddle points

Games With no Saddle Points

		Player 2		
		A	B	C
Player 1	A	2	0	-1
	B	-5	3	1

- What should players do?
 - resort to randomness to select strategies

Mixed Strategies

- Each player associates a probability distribution over its set of strategies
 - players decide on which prob. distribution to use
- Payoffs are computed as expectations

		1/3	2/3
		C	D
Player 1	A	2	0
	B	-5	3

Payoff to P1 when playing A = $1/3(2) + 2/3(0) = 2/3$

Payoff to P1 when playing B = $1/3(-5) + 2/3(3) = 1/3$

- How should players choose prob. distribution?

Mixed Strategies

- Idea: use a prob. distribution that cannot be exploited by other player
 - payoff should be equal independent of the choice of strategy of other player
 - guarantees minimum gain (maximum loss)
- How should Player 2 play?

		x	(1-x)
		C	D
Player 1	A	2	0
	B	-5	3

Payoff to P1 when playing A = $x(2) + (1-x)(0) = 2x$

Payoff to P1 when playing B = $x(-5) + (1-x)(3) = 3 - 8x$

$2x = 3 - 8x$, thus $x = 3/10$

Mixed Strategies

- Player 2 mixed strategy
 - 3/10 C , 7/10 D
 - maximizes its loss independent of P1 choices
- Player 1 has same reasoning

		Player 2		
		C	D	
Player 1	x	A	2	0
	(1-x)	B	-5	3

Payoff to P2 when playing C = $x(-2) + (1-x)(5) = 5 - 7x$

Payoff to P2 when playing D = $x(0) + (1-x)(-3) = -3 + 3x$

$5 - 7x = -3 + 3x$, thus $x = 8/10$

Payoff to P1 = 6/10

Minimax Theorem

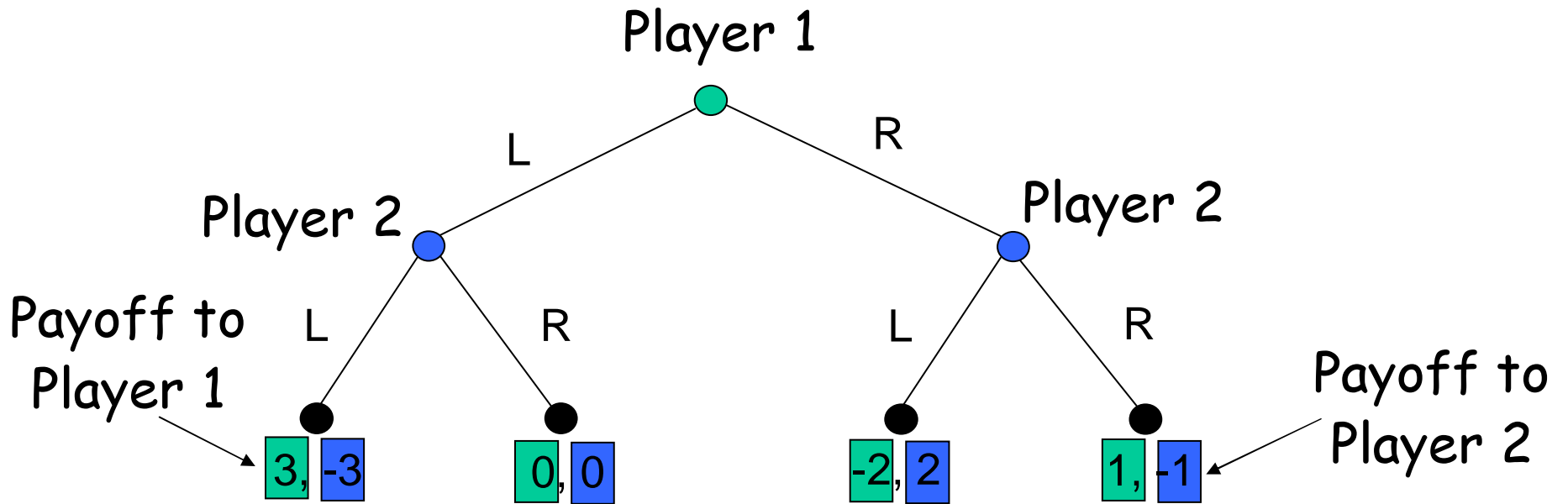
- Every two-person zero-sum game has a solution in mixed (and sometimes pure) strategies
 - solution payoff is the value of the game
 - maximin = v = minimax
 - v is unique
 - multiple equilibrium in pure strategies possible
 - but fully interchangeable

- Proved by John von Neumann in 1928!
 - birth of game theory...

Game Trees (Extensive form)

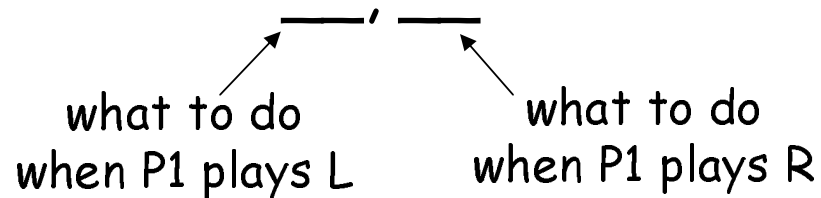
- Sequential play
 - players take turns in making choices
 - previous choices are available to players
- Game represented as a tree
 - each non-leaf node represents a decision point for some player
 - edges represent available choices
- Can be converted to matrix game (Normal form)
 - “plan of action” must be chosen before hand

Game Trees Example



- Strategy set for Player 1: $\{L, R\}$

- Strategy for Player 2:

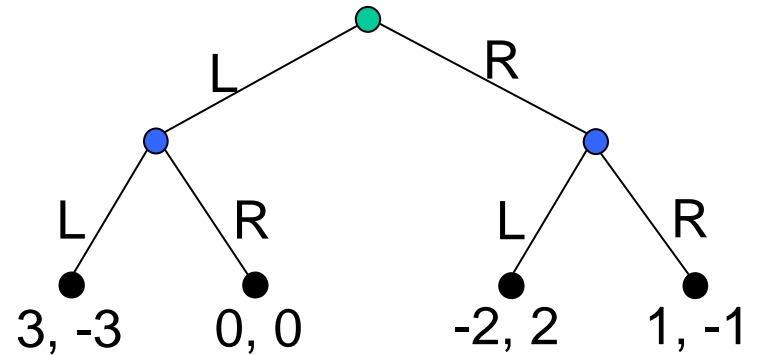


- Strategy set for Player 2: $\{LL, LR, RL, RR\}$

More Formal Extensive Game Definition

- An extensive form game
 - a finite set N of players
 - a finite height game tree
 - payoff function $u_i(s)$ for each player $i \in N$
 - where s is a leaf node of game tree
- Game tree: set of nodes and edges
 - each non-leaf node represents a decision point for some player
 - edges represent available choices (possibly infinite)
- Perfect information
 - all players have full knowledge of game history

Converting to Matrix Game

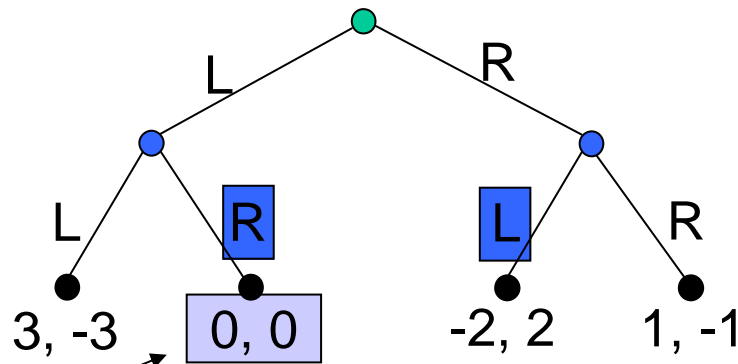


		Player 2			
		LL	LR	RL	RR
Player 1	L	3	3	0	0
	R	-2	1	-2	1

- Every game in extensive form can be converted into normal form
 - exponential growth in number of strategies

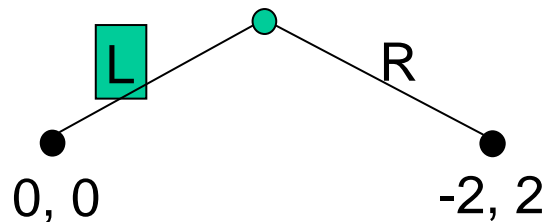
Solving the Game (backward induction)

- Starting from terminal nodes
 - move up game tree making best choice



Best strategy
for P2: RL

Equilibrium
outcome



Best strategy
for P1: L

- Saddle point:
P1 chooses L, P2 chooses RL

Kuhn's Theorem

- Backward induction always leads to saddle point (on games with perfect information)
 - game value at equilibrium is unique (for zero-sum games)

- Consider a modified game of chess
 - either white wins (+1, -1)
 - either black wins (-1, +1)

- Backward induction on game tree
 - white has winning strategy no matter what black does
 - black has winning strategy no matter what white does

Chess is a simple game!

Two-person Non-zero Sum Games

- Players are not strictly opposed
 - payoff sum is non-zero

		Player 2	
		A	B
Player 1	A	3, 4	2, 0
	B	5, 1	-1, 2

- Situations where interest is not directly opposed
 - players could cooperate

What is the Solution?

- Ideas of zero-sum game: saddle points
- pure strategy equilibrium
- mixed strategies equilibrium
 - no pure strat. eq.

		Player 2	
		A	B
Player 1	A	5, 4	2, 0
	B	3, 1	-1, 2

		Player 2	
		A	B
Player 1	A	5, 0	-1, 4
	B	3, 2	2, 1

Multiple Solution Problem

- Games can have multiple equilibria
 - not equivalent: payoff is different
 - not interchangeable: playing an equilibrium strategy does not lead to equilibrium

		Player 2	
		A	B
Player 1	A	1, 4	1, 1
	B	0, 1	2, 2

equilibria

The Good News: Nash's Theorem

- Every two person game has *at least one* equilibrium in either pure or mixed strategies
- Proved by Nash in 1950 using fixed point theorem
 - generalized to N person game
 - did not "invent" this equilibrium concept
- Def: An outcome o^* of a game is a NEP (Nash equilibrium point) if no player can unilaterally change its strategy and increase its payoff
- Cor: any saddle point is also a NEP

The Prisoner's Dilemma

- One of the most studied and used games
 - proposed in 1950
- Two suspects arrested for joint crime
 - each suspect when interrogated separately, has option to confess

		Suspect 2	
		Q	C
Suspect 1	Q	2, 2	10, 1
	C	1, 10	5, 5

payoff is years in jail (smaller is better)


better outcome

single NEP

Pareto Optimal

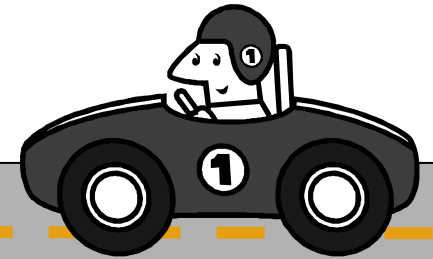
- Prisoner's dilemma: individual rationality

		Suspect 2	
		Q	C
Suspect 1	Q	2, 2	10, 1
	C	1, 10	5, 5

 Pareto Optimal

- Another type of solution: group rationality
 - Pareto optimal
- Def: outcome o^* is Pareto Optimal if no other outcome is better for *all* players

Game of Chicken Revisited



- Game of Chicken (aka. Hawk-Dove Game)
 - driver who swerves loses

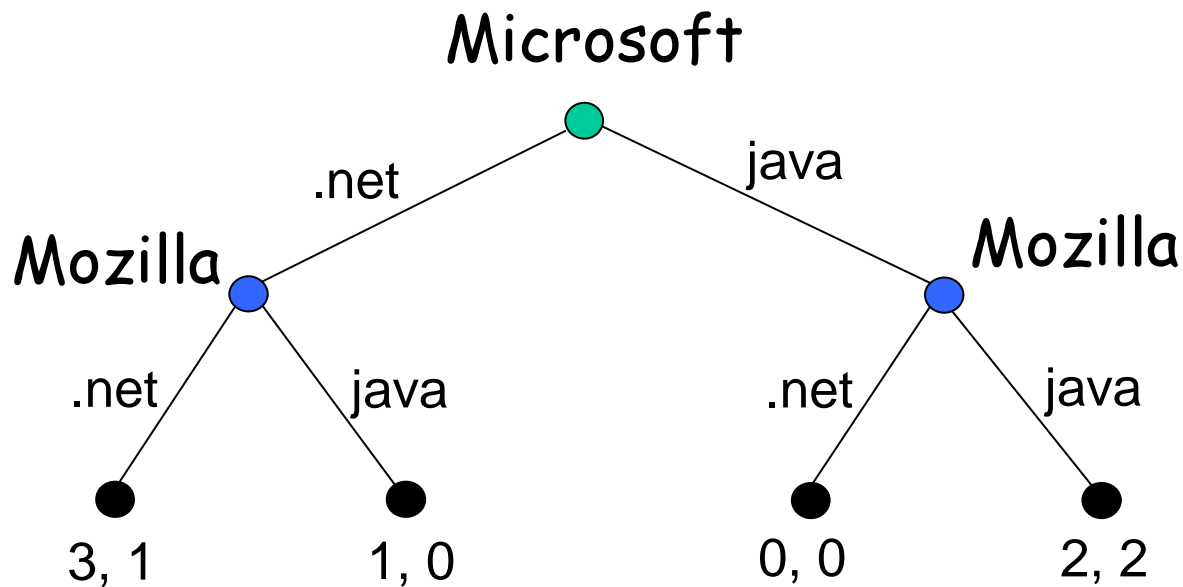
		Driver 2	
		swerve	stay
Driver 1	swerve	0, 0	-1, 5
	stay	5, -1	-10, -10

Drivers want to do opposite of one another

Will prior communication help?

Game Trees Revisited

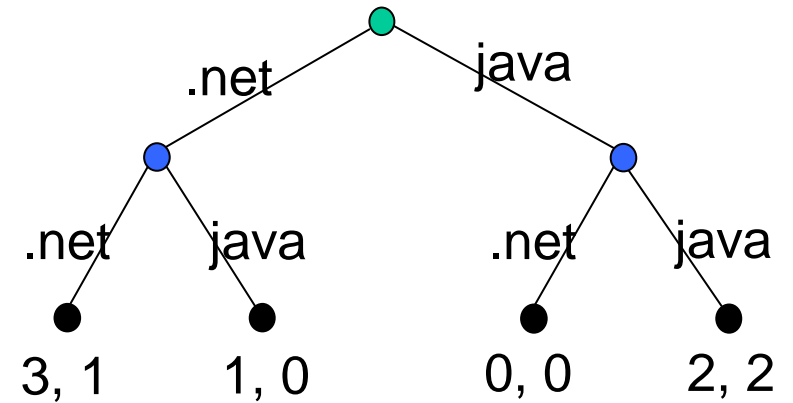
- Microsoft and Mozilla are deciding on adopting new browser technology (.net or java)
 - Microsoft moves first, then Mozilla makes its move



- Non-zero sum game
 - what are the NEP?

NEP and Incredible Threats

- Convert the game to normal form



		Mozilla			
		NN	NJ	JN	JJ
Microsoft	.net	3,1	3,1	1,0	1,0
	java	0,0	2,2	0,0	2,2

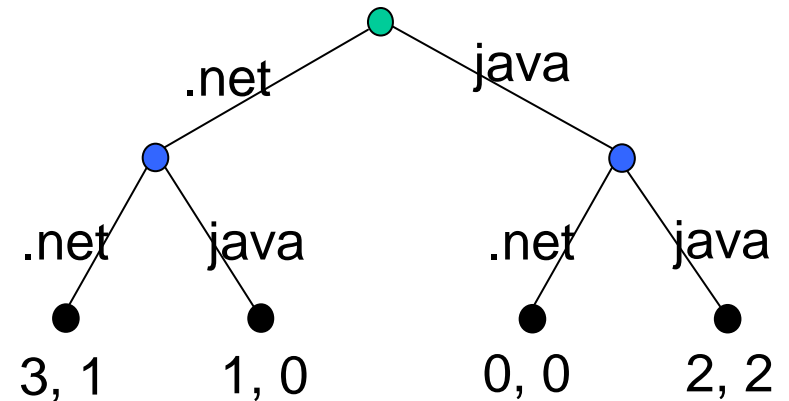
NEP

incredible threat

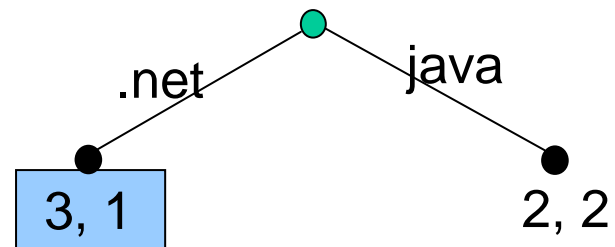
- Play "java no matter what" is not credible for Mozilla
 - if Microsoft plays .net then .net is better for Mozilla than java

Removing Incredible Threats and other poor NEP

- Apply backward induction to game tree



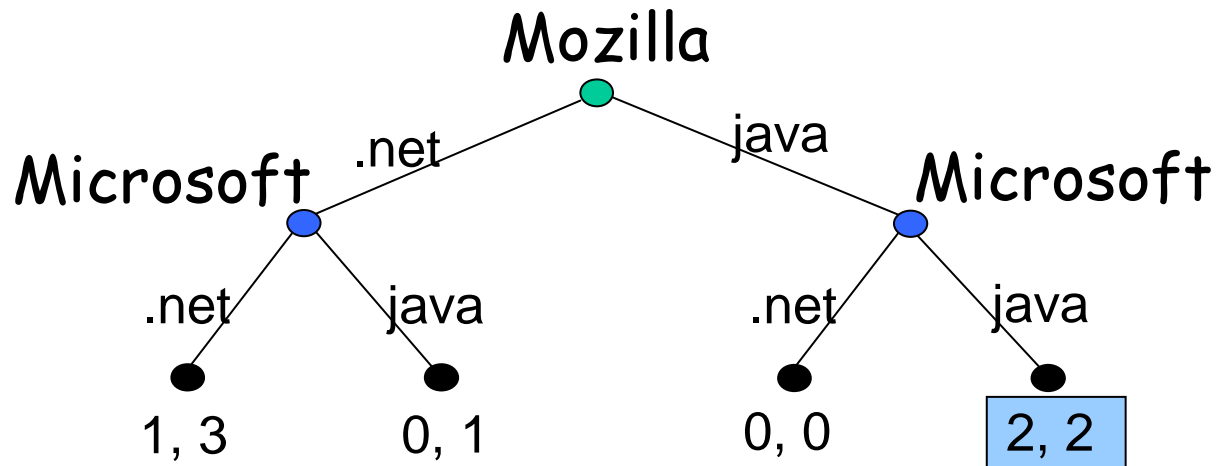
- Single NEP remains
.net for Microsoft,
.net, java for Mozilla



- In general, multiple NEPs are possible after backward induction
 - cases with no strict preference over payoffs

Leaders and Followers

- What happens if Mozilla moves first?



- NEP after backward induction: Mozilla: java
Microsoft: .net, java
- Outcome is better for Mozilla, worst for Microsoft
 - incredible threat becomes credible!
- 1st mover advantage
 - but can also be a disadvantage...

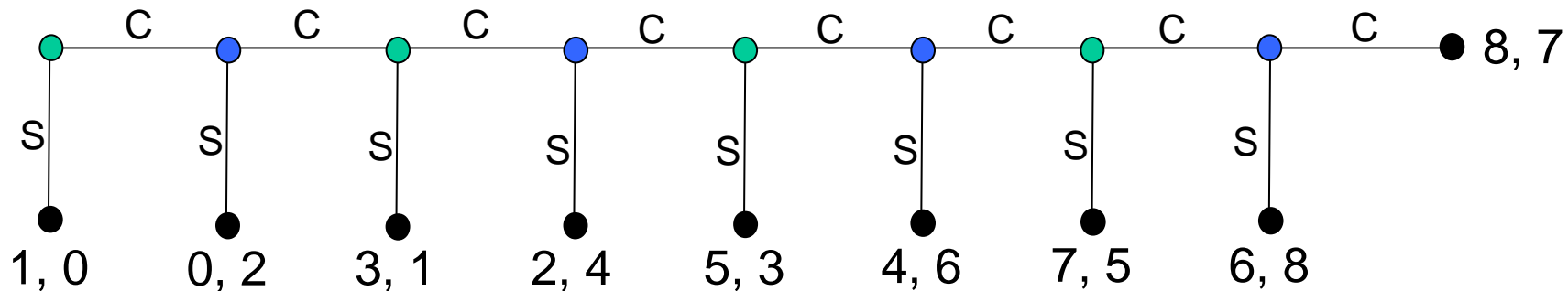
Subgame Perfect Nash Equilibrium

- Set of NEP that survive backward induction
 - in games with perfect information
- Def: a subgame is any subtree of the original game that also defines a proper game
- Def: a NEP is *subgame perfect* if its restriction to *every* subgame is also a NEP of subgame
- Thr: every extensive form game with complete information has at least one subgame perfect Nash equilibrium
 - Kuhn's theorem, based on backward induction

Weakness of SPNE

□ Centipede Game

- two players alternate decision to continue or stop for k rounds
- stopping gives better payoff than next player stopping in next round (but not if next player continues)



- Backward induction leads to unique SPNE
 - both players choose S in every turn
- Each player believes that the other player will stop the game in next opportunity
- How would you play this game with a stranger?
 - empirical evidence suggests people continue for many rounds

Stackelberg Game

- ❑ Two players, leader and follower
- ❑ Two moves, leader then follower
 - can be modeled by a game tree

- ❑ Stackelberg equilibrium
 - Leader chooses strategy knowing that follower will apply best response
 - this precludes incredible threats
- ❑ Similar to subgame perfect Nash equilibrium
 - every Stackelberg equilibrium is also SPNE

Repeated Games

- Game played an *indefinite* number of times
 - same game, same set of players
- Important model in practice
 - many scenarios repeat themselves
- Anomalies of finitely repeated games disappear
 - cooperation can sometimes emerge!

Details to follow in next lecture...