Efficiency in loss in a Network Resource Allocation Game

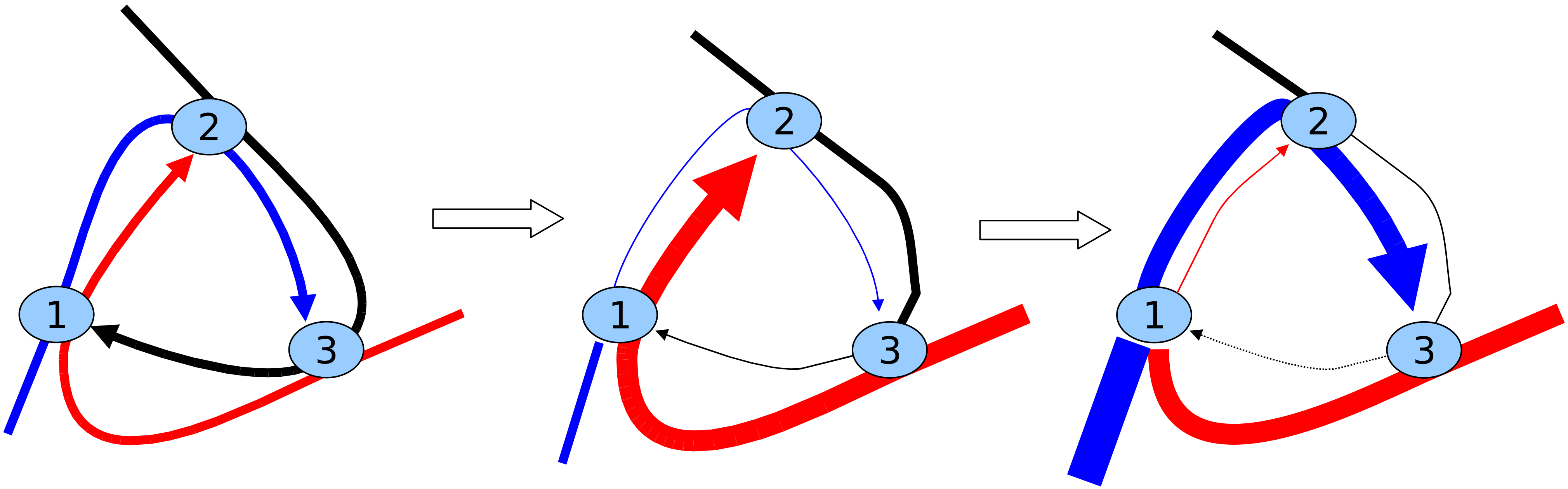
Ramesh Johari, John N. Tsitsiklis

Mathematics of Operations Research 29 (3): 407-435

Presentation/Presenter: Bruno Ribeiro

100% total anarchy

- Do users sharing resources need to behave?



Efficient allocation of resources

- Let d_r be consumer's ***r*** demand
- Let $U_r(d_r)$ be the consumer's ***r*** utility function associated to demand d_r.
- Pareto efficient allocation
  - One cannot increase $U_r(d_r)$ without decreasing some $U_k(d_k)$.

How to achieve Pareto efficiency in a market?

Banana market

- UDS (a producer) is giving away S bananas
- Two students (consumers) are competing for them
- Any solution to the SYSTEM problem

$$\begin{aligned} \text{maximize} \quad & U_1(d_1) + U_2(d_2) \\ \text{subject to} \quad & d_1 + d_2 \leq S; \\ & d_1, d_2 \geq 0. \end{aligned}$$

is Pareto efficient.

Market-clearing producers

Market-clearing producers

- Producers want to sell all their resources
  - Price is set to clear the stock

- Are Internet routers market-clearing?
  - Not really....
  - We will see a better approximation later

Price taking consumers

- Producer has to define a price p^*
  - Suppose no price discrimination
- Let $D_r(p^*)$ be a demand function of consumer r
  - With price p^*, consumer will buy $D_r(p^*)$
- The consumer payoff

w_r = bid

$$U_r(D_r(p^*)) - p^* D_r(p^*)$$

OR

$$P_r(w_r; p^*) = U_r(w_r / p^*) - w_r$$

Competitive equilibrium

- Consumer r chooses a bid w_r that

$$\max \quad P_r(w_r; p^*)$$

maximizes its individual payoff, given a market price.

- Producer price must clear the stock:

$$\sum_r D_r(p^*) = C$$

C is the producer's supply

- Or $\sum_r w_r / p^* = C$ ⇒ $p^* = \sum_r w_r / C$

What is a price-taking consumer?

- Consumers cannot anticipate the effect of their bid on the market-clearing price.
- Selfish consumers translates into:
  - Pareto efficient solution
    or the optimal of the sum of utilities (F. Kelly 97)
- What if they could anticipate the effect?

Price anticipating consumers

- Now consumers won't assume p^* to be fixed

$$P_r(w_r; \boldsymbol{w}_{-r}) = U_r(w_r C / (\sum \boldsymbol{w}_{-r} + w_r)) - w_r$$

- And the payoff maximization is find w_r s.t.

$$\max \quad U_r(w_r C / (\sum \boldsymbol{w}_{-r} + w_r)) - w_r$$

- Now consumer r payoff depends on the choice made by other consumers ($\boldsymbol{w}_{-r}$).
  - Thus defining a game

One extra thing

- We will still talk about $U_r(.)$ because we will mostly see examples where $w_r > 0$.
- If $w_r = 0$ is allowed, then we need a new utility

maximize:

$$Q_r(w_r; \mathbf{w}_{-r}) = \begin{cases} U_r\left(\frac{w_r}{\sum_s w_s} C\right) - w_r, & \text{if } w_r > 0; \\ U_r(0), & \text{if } w_r = 0. \end{cases}$$

- Notice that discontinuity in function $Q_r(.)$

An example with two players

- $C = 1$
- $U_1(w_1 / (w_1 + w_2)) = \alpha_1 w_1 / (w_1 + w_2)$
- $U_2(w_2 / (w_1 + w_2)) = \alpha_2 w_2 / (w_1 + w_2)$
- Consumer 1 payoff maximization is

$$\max_{w1 \geq 0} [\alpha_1 w_1 / (w_1 + w_2) - w_1]$$

- Notice that payoff of consumer 1 is concave in w_1
- Assuming $w_1, w_2 > 0$ ....

Nash equilibrium

- Selfish consumers ⇨ maximize their payoffs
- Concave function:
  First derivative is zero at the optimal point

$$\alpha_1 \left(\frac{1}{w_1 + w_2} - \frac{w_1}{(w_1 + w_2)^2} \right) = 1$$

$$\alpha_2 \left(\frac{1}{w_1 + w_2} - \frac{w_2}{(w_1 + w_2)^2} \right) = 1$$

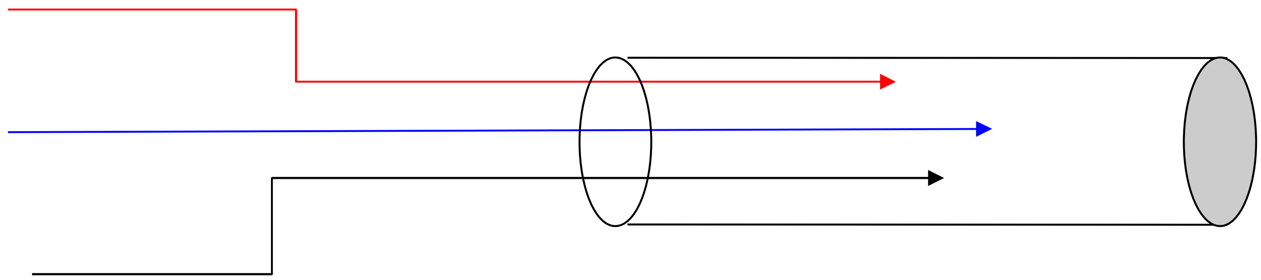
The <u>unique</u> solution to the above system is a NE point

$$w_1^{NE} = \frac{\alpha_1^2 \alpha_2}{(\alpha_1 + \alpha_2)^2}; \qquad w_2^{NE} = \frac{\alpha_1 \alpha_2^2}{(\alpha_1 + \alpha_2)^2}$$

Single link

Model

R flows or R consumers



Channel with capacity C
or Producer with supply C

Single link

- If the utility function U_r is concave, strictly increasing and continuously differentiable
- More than one player ($R > 1$)
- Then the NE point exists and is unique (Theorem 2, Hajek and Gopalakrishnan)

Price of anarchy

- Assume $\alpha_1 > \alpha_2 > 0$
- Pareto efficient allocation:
  - $d^*_1 = C$; $d^*_2 = 0$
- Price of anarchy:

$$\frac{\text{NE sum of utilities}}{\text{Optimal sum of utilities}}$$

In the example:

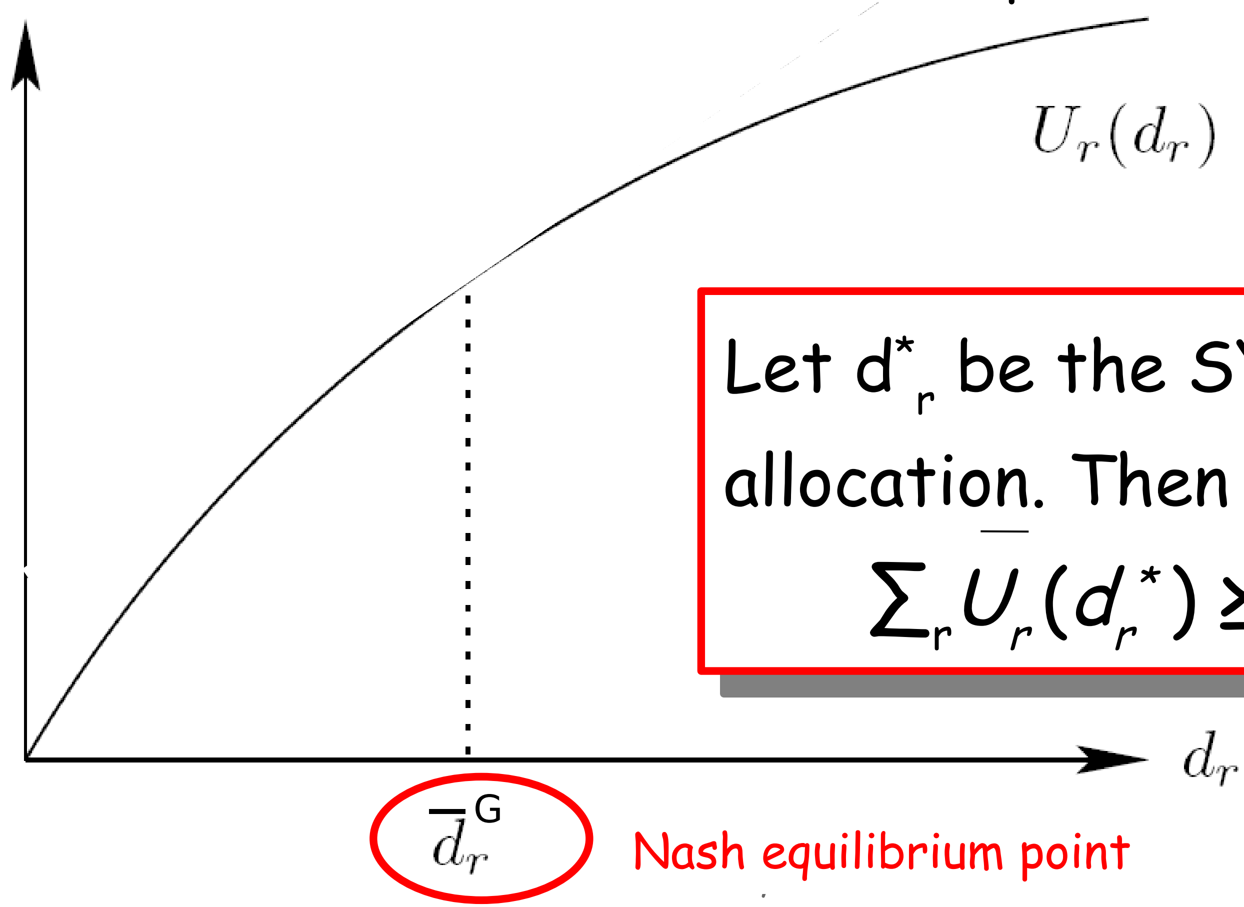
$$\frac{\alpha_1 d_1^{NE} + \alpha_2 d_2^{NE}}{\alpha_1 d_1^* + \alpha_2 d_2^*} = \frac{\alpha_1^2 + \alpha_2^2}{\alpha_1^2 + \alpha_1 \alpha_2}$$

Price of anarchy, single link

"Worst efficiency loss happens when utility is linear"

Reminder: Utility $U_r()$ is concave

$\overline{U}_r(d_r)$: Linear replacement for $U_r(d_r)$



Let d^*_r be the SYSTEM optimal allocation. Then

$$\sum_r \overline{U}_r(d_r^*) \geq \sum_r U_r(d_r^*)$$

Price of anarchy, single link

Accessing the price of anarchy

We can easily derive the NE/(SYSTEM OPTIMAL) ratio for linear utility functions

$$\frac{\sum_r \alpha_r d_r^G}{(\max_r \alpha_r)\, C}$$

Reescale the problem such that C=1 and $\max_r \alpha_r = 1$

It remains to find a NE that has the worst aggregate utility under C=1 and $\max_r \alpha_r = 1$.

Price of anarchy, single link

- Without loss of generality assume:
  - $C = 1$ AND $\max_r \alpha_r = \alpha_1 = 1$

Goal: Find $\alpha_2,\ldots,\alpha_R$ such that the total utility of the Nash equilibrium

$$d_1^G + \sum_{r=2}^{R} \alpha_r d_r^G$$

is as small as possible.

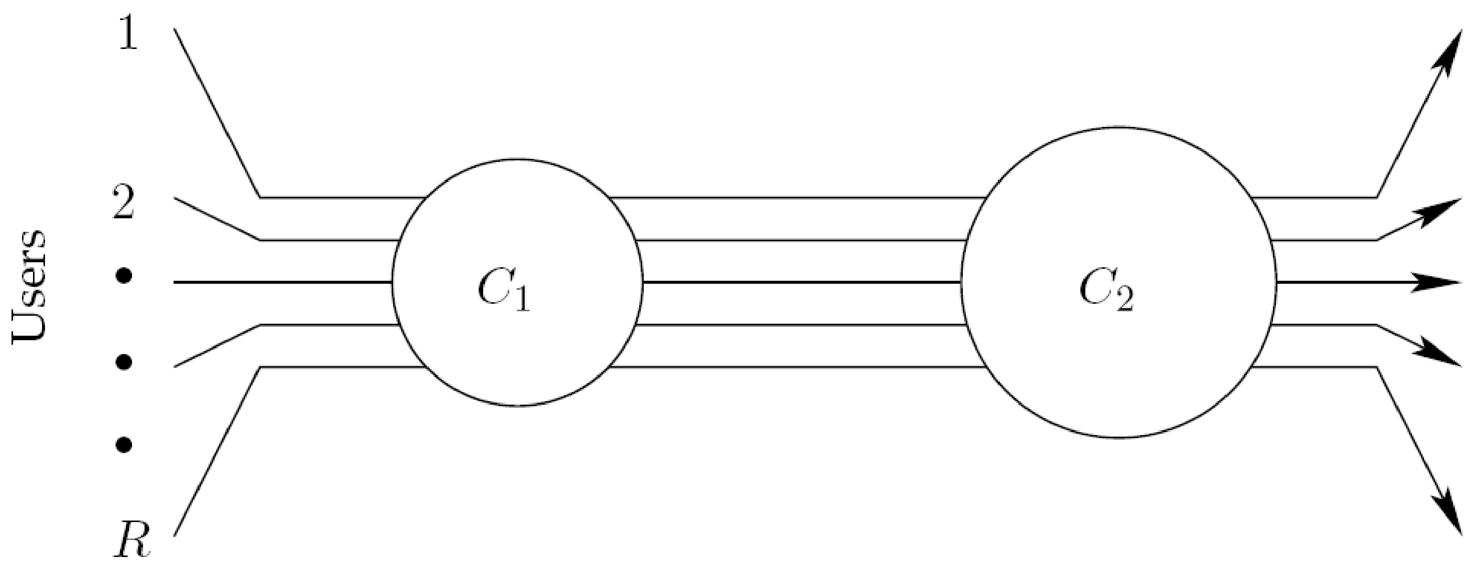
Price of anarchy, single link

- The previous can be thought as an optimization problem by itself
  - There is a NE only when $d_r^G \geq 1/R$
  - The smallest sum of the utilities is 3/4

Multiple links

- Now we will look at results using multiple links

- Don't be alarmed with the link index "j"



Problems with market-clearing

- Any fundamental issue with the previous model?
- All capacity must be allocated, even if consumers can't / don't want to use it

Slight (extended) market change

- Consumer r submits a bid w_r and a rate request
- The strategy of consumer r is a vector
  - $\sigma_r = (\varphi_r, w_r)$
- Addresses the problem with
  - If $\sum_r w_{j,r} = 0$ and $\sum_r \varphi_{j,r} \leq C_j \Rightarrow d_{j,r} = \varphi_{j,r}$
  - If $\sum_r w_{j,r} = 0$ and $\sum_r \varphi_{j,r} > C_j \Rightarrow d_{j,r} = 0$

Slight market change

- Payoff function looks like

$$T_r(\boldsymbol{\sigma}_r; \boldsymbol{\sigma}_{-r}) = U_r\Big(d_r(\mathbf{x}_r(\boldsymbol{\sigma}))\Big) - \sum_j w_{jr}$$

where

$$\mathbf{x}_r(\boldsymbol{\sigma}) = (x_{jr}(\boldsymbol{\sigma}), j \in J)$$

and

$$d_r(\mathbf{x}_r(\mathbf{w})) = \min\{x_{1r}(\mathbf{w}), x_{2r}(\mathbf{w})\}$$

NE of this new market

- Always exists, even for $w_{j,r} = 0$ (Theorem 6)
  - Actually, the rate request works to clean the market when $\sum_r w_{j,r} = 0$, which was our previous concern.

- Uniqueness is an open question

Price of anarchy of the new market

- As previously, the price of anarchy is 3/4 (Theorem 7)
  - Map the previous straight line to an hyperplane?

Final considerations

- Is bandwidth allocation market-clearing?
- Doesn't match Kelly's framework
  - Author says it approximates
  - Due to the rate-request “fix”

Sources

- Ramesh Johari – "Efficiency Loss in Market Mechanisms for Resource Allocation" – Ph.D. Thesis, Massachusetts Institute of Technology
- R. Johari and J. N. Tsitsiklis - "Efficiency Loss in a Network for Resource Allocation Game" - Mathematics of Operations Research 29 (3): 407-435.