**cx, icx Stochastic Orderings**

We focus only on nonnegative valued random variables.

**Defn.** Let $X, Y \in (\mathbb{R}^+)^n$ be random variables such that $E[f(X)] \geq E[f(Y)] \ \forall f$ that are cx (resp. icx). Then we say that $X$ is larger than $Y$ in the sense of convex (resp. increasing convex) order and write it as $Y \preceq_{cx} X$ (resp. $Y \preceq_{icx} X$).
Properties of \( cx, icx \) Orderings

Let \( X, Y \in \Re^+ \). If \( X \geq_{cx} Y \), then

- \( E[X] = E[Y] \),
- \( \sigma_X^2 \geq \sigma_Y^2 \),
- \( E[X^k] \geq E[Y^k] \), \( k > 1 \).

If \( X \geq_{icx} Y \), then \( E[X^k] \geq E[Y^k] \), \( k \geq 1 \). Furthermore, if \( E[X] = E[Y] \), then \( X \geq_{cx} Y \).

Last, if \( Y \) is a constant, then \( Y \leq_{cx} X \) \( \forall X \) st \( E[X] = Y \).
**G/G/1 Queue**

Single server queue with FCFS (first come first serve). Let \( \{\tau_n\} \) and \( \{X_n\} \) be mutually independent iid sequences of rvs corresponding to interarrival times and service times respectively. Let \( \{W_n\} \) denote wait times for customers. We have

\[
W_0 = 0 \\
W_{n+1} = \max(0, W_n + X_n - \tau_{n+1})
\]
Ordering for $G/G/1$ Queue

**Thm.** If $\tau_n^{(1)} \geq_{cx} \tau_n^{(2)}$, $X_n^{(2)} \leq_{icx} X_n^{(1)}$, $\forall n$, then $W_n^{(2)} \leq_{icx} W_n^{(1)}$, $\forall n$.

**Proof.** By induction.
NBUE, NWUE Random Variables

**Defn.** $X \in \mathbb{R}^+$ is said to be *new better than used in expectation* (NBUE) iff

$$E[X - a | X > a] \leq E[X], \quad \forall a > 0$$

$X \in \mathbb{R}^+$ is said to be *new worse than used in expectation* (NBUE) iff

$$E[X - a | X > a] \geq E[X], \quad \forall a > 0$$

**Examples.**

- an exponential rv $X$ is both NBUE and NWUE
- an Erlang rv $X$ of order $r > 1$ is NBUE
- a deterministic rv is NBUE
- an $H_r$ rv $X$ is NWUE

**Thm.** Suppose that $X$ is NBUE, $Y$ is an exponential rv, and $E[X] = E[Y]$, then $X \leq_{cx} Y$. If $X$ is NWUE, then $X \geq_{cx} Y$. 

5
G/G/1 with NBUE, NWUE Interarrival Times

Consider a G/G/1 queue that exhibits steady state ($\lambda E[X] < 1$). Suppose that the interarrival times are NBUE with mean $1/\lambda$, then

$$E[W] \leq \frac{\lambda E[X^2]}{2(1 - \lambda E[X])}$$

If the interarrival times are NWUE, then

$$E[W] \geq \frac{\lambda E[X^2]}{2(1 - \lambda E[X])}$$

These are come from the icx comparison results coupled with the comparisons between NBUE, NWUE rvs and exponential rvs.
Comparison of scheduling policies for $G/G/1$ queue

- $G/G/1$ queue, interarrival times $\{\tau_n\}$, iid service times $\{X_n\}$. $\{X_n\}$ independent of $\{\tau_n\}$.
- $\Sigma_{np}$ - class of work conserving (non-idling) non-preemptive policies that do not use service time information. e.g., FIFO, LIFO $\in \Sigma_{np}$
- assume steady state; $S_{\pi}$ - steady state sojourn time under policy $\pi \in \Sigma_{np}$
- note: $E[S_{\pi}]$ does not depend on $\pi$

**Thm.** FIFO minimizes $S_{\pi}$ in the sense of cx order, LIFO maximizes $S_{\pi}$ in the sense of cx order,

$$S_{FIFO} \leq_{cx} S_{\pi} \leq_{cx} S_{LIFO}, \quad \forall \pi \in \Sigma_{np}$$
Comparison of scheduling policies for $G/G/1$ queue

Proof consists of 2 parts. First restrict ourselves to $N$ arrivals. Second take limit as $N \to \infty$. Define cost function

$$C(N, \pi) = \sum_{i=1}^{N} f(S_i(\pi)),$$

$f(\cdot)$ a cx function, $S_i(\pi)$ sojourn time of $i$th job under $\pi$. We are interested in $E[C(N, \pi)]$.

iid service times implies that they can be assigned in order of service. Condition on arrival times $a_1 < a_2 < \cdots < a_N$ and service times $x_1, x_2, \ldots, x_N$.

Number jobs in order of arrival. $\pi_m$ is index of job scheduled in $m$-th position by $\pi$, $m = 1, \ldots, N$.

Let $d_l$ denote $l$-th departure time; does not depend on $\pi$.

$$d_l = \max(d_{l-1}, a_l) + s_l$$

$$d_0 = 0$$
Comparison of scheduling policies for $G/G/1$ queue

Consider arbitrary policy $\pi \in \Sigma_{np}$, $\pi \neq FIFO$. Let $\pi$ differ from FIFO for first time at $m$-th scheduling decision,

$$
\begin{align*}
\pi_m &= u \neq m \\
\pi_l &= l & l = 1, \ldots, m-1 \\
\pi_v &= m & v > m
\end{align*}
$$

Construct policy $\pi'$ that behaves like $\pi$ for all but scheduling decisions $m, v$.

$$
\begin{align*}
\pi'_l &= \pi_l & l \neq m, v \\
\pi'_m &= m \\
\pi'_v &= u
\end{align*}
$$

See figure on next page.
Comparison of scheduling policies for $G/G/1$ queue
Comparison of scheduling policies for $G/G/1$ queue

Sum of response times the same for $\pi$ and $\pi'$. Difference in cost function is

$$C(N, \pi) - C(N, \pi')$$

$$= f(S_m(\pi)) + f(S_u(\pi)) - [f(S_m(\pi)) + f(S_u(\pi))]$$

$$= [f(d_v - a_m) + f(d_m - a_u)]$$

$$- [f(d_v - a_u) + f(d_m - a_m)]$$

$$\geq 0$$

Other terms are equal and cancel. Inequality is consequence of convexity of $f$.

Repeat process on $\pi'$ to obtain series of policies $\pi, \pi', \pi'', \ldots$. After finite number of steps, this process generates FIFO. This produces

$$C(N, \pi) \geq C(N, \pi') \geq C(N, \pi'') \geq \cdots \geq C(N, FIFO)$$
Comparison of scheduling policies for $G/G/1$ queue

Remove conditioning on arrival times and service times to yield

$$E[C(N, \pi)] \geq E[C(N, FIFO)]$$

in limit as $N \to \infty$,

$$\frac{1}{N}E[C(N, \pi)] \to E[f(S_\pi)]$$

Therefore,

$$S_\pi \geq_{cx} S_{FIFO}$$

A similar argument yields

$$S_\pi \leq_{cx} S_{LIFO}$$