More General Systems

- $G/G/1$ queue
  - matrix geometric techniques
  - other matrix techniques
  - bounds

- other service disciplines
  - priority queueing
  - last come first serve without preemptions (LCFS); with preemptions LCSFPR
  - processor sharing
$M/M/1$ Queue Revisited

- one minor modification, arrival rate $\lambda'$ when none in system; $\lambda$ otherwise
- infinitesimal generator $Q$

\[
Q = \begin{bmatrix}
-\lambda' & \lambda' & 0 & 0 & \cdots \\
\mu & -(\lambda + \mu) & \lambda & 0 & \cdots \\
0 & \mu & -(\lambda + \mu) & \lambda & \cdots \\
0 & 0 & \mu & -(\lambda + \mu) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

- balance equations:
  
  \[
  -\pi_0 \lambda' + \pi_1 \mu = 0 \\
  \pi_0 \lambda' - \pi_1 (\lambda + \mu) + \pi_2 \mu = 0 \\
  \pi_{j-1} \lambda - \pi_j (\lambda + \mu) + \pi_{j+1} \mu = 0, \quad j = 2, 3, \ldots
  \]

- guess $\pi_j = \pi_1 r^{j-1}$ implies

  \[
  \pi_1 r^j \mu - \pi_1 r^{j-1} (\lambda + \mu) + \pi_1 r^{j-2} \lambda = 0
  \]

  or

  \[
  r^2 \mu - r(\lambda + \mu) + \lambda = 0
  \]
**M/M/1 Queue cont.**

Notice that there are two solutions to the last equation, \( r = \lambda/\mu, 1 \). The solution \( r = 1 \) does not make sense since 
\[
\sum_{j=1}^{\infty} \pi_j = \infty
\]
Therefore, when \( \lambda/\mu < 1 \), the solution \( r = \lambda/\mu \) makes sense, and \( \pi_j = \pi_1 \rho^{j-1} \) (where \( \rho \equiv \lambda/\mu \)).

**Q:** how to obtain \( \pi_0 \) and \( \pi_1 \)?

\[
(\pi_0, \pi_1) \begin{bmatrix} -\lambda' & \lambda' \\ \lambda' & -(\lambda + \mu) + \rho \mu \end{bmatrix} = [0 \ 0]
\]

along with

\[
1 = \pi_0 + \pi_1 \sum_{j=1}^{\infty} \rho^{j-1}
\]
\[
= \pi_0 + \pi_1/(1 - \rho)
\]

Finally, \( \pi_0 \) and \( \pi_1 \) are solutions to

\[
(\pi_0, \pi_1) \begin{bmatrix} 1 & \lambda' \\ 1/(1 - \rho) & -(\lambda + \mu) + \rho \lambda \end{bmatrix} = [1 \ 0]
\]

**Q:** what is \( E[N_q] \)?

\[
E[N_q] = \sum_{j=1}^{\infty} (j - 1)\pi_j = \pi_1 \sum_{j=1}^{\infty} (j - 1)\rho^{j-1}
\]
\[
= \pi_1 \rho(1 - \rho)^{-2}
\]
Hyperexponential Distribution \((H_r)\)

- An rv \(X\) with an \(r\) stage *hyperexponential* distribution \((H_r)\) distr. has following density function

\[
f_X(t) = \sum_{i=1}^{r} \alpha_i \mu_i e^{-\mu_i t}, \quad t \geq 0
\]

\((\sum_{i=1}^{r} \alpha_i = 1)\) with mean and coeff. of variation

\[
E[X] = \sum_{i=1}^{r} \frac{\alpha_i}{\mu_i}
\]

\[
C_X^2 = \frac{2 \sum_{i=1}^{r} \frac{\alpha_i}{\mu_i^2}}{\left(\sum_{i=1}^{r} \frac{\alpha_i}{\mu_i}\right)^2} - 1
\]

Can show \(C_X^2 \geq 1\) using Cauchy-Schwarz inequality
$M/H_2/1$ Queue

- Poisson arrival process, $\lambda'$ when no jobs, $\lambda$ when one or more jobs
- service times given by $H_2$ distr. with parameters $\alpha, \mu_1$; $\bar{\alpha}, \mu_2$
- system state - $(n, s)$
  - $n$ no. in system
  - $s$ exponential stage of job in service
  - when $n = 0$, $s = 0$
- state transition diagram
$M/H_2/1$ Queue

Infinitesimal generator $Q$

$$Q = \begin{bmatrix}
  -\lambda' & \lambda'\alpha & \lambda'\bar{\alpha} & 0 & 0 & 0 & 0 & 0 & \cdots \\
  \mu_1 & -a_1 & 0 & \lambda & 0 & 0 & 0 & 0 & \cdots \\
  \mu_2 & 0 & -a_2 & 0 & \lambda & 0 & 0 & 0 & \cdots \\
  0 & \alpha\mu_1 & \bar{\alpha}\mu_1 & -a_1 & 0 & \lambda & 0 & \cdots \\
  0 & \alpha\mu_2 & \bar{\alpha}\mu_2 & 0 & -a_2 & 0 & \lambda & \cdots \\
  0 & 0 & 0 & \alpha\mu_1 & \bar{\alpha}\mu_1 & -a_1 & 0 & \cdots \\
  0 & 0 & 0 & \alpha\mu_2 & \bar{\alpha}\mu_2 & 0 & -a_2 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}$$

where $a_i = \lambda + \mu_i$, $i = 1, 2$

Define the following submatrices

$$B_{0,0} \equiv [-\lambda'], \quad B_{0,1} \equiv [\lambda'\alpha \; \lambda'\bar{\alpha}], \quad B_{1,0} \equiv \begin{bmatrix} \mu_1 \\
\mu_2 \end{bmatrix}$$

$$A_0 \equiv \begin{bmatrix} \lambda & 0 \\
0 & \lambda \end{bmatrix}, \quad A_2 \equiv \begin{bmatrix} \alpha\mu_1 & \bar{\alpha}\mu_1 \\
\alpha\mu_2 & \bar{\alpha}\mu_2 \end{bmatrix},$$

$$A_1 \equiv \begin{bmatrix} -(\lambda + \mu_1) & 0 \\
0 & -(\lambda + \mu_2) \end{bmatrix}$$
$M/H_2/1$ Queue

Let $\pi_i = (\pi_{i,1}, \pi_{i,2})$, $i = 1, 2, \ldots$ and $\pi_0 = (\pi_{0,0})$. Rewrite the infinitesimal generator $Q$ as

$$Q = \begin{bmatrix}
B_{0,0} & B_{0,1} & 0 & 0 & 0 & \cdots \\
B_{1,0} & A_1 & A_0 & 0 & 0 & \cdots \\
0 & A_2 & A_1 & A_0 & 0 & \cdots \\
0 & 0 & A_2 & A_1 & A_0 & \cdots \\
: & : & : & : & : & : 
\end{bmatrix}$$

This is equivalent to

$$\pi_{j-1}A_0 + \pi_j A_1 + \pi_{j+1} A_2 = 0, \quad j = 2, 3, \ldots$$

Let us conjecture the existence of matrix $R$ such that

$$\pi_j = \pi_1 R^{j-1}, \quad j = 1, \ldots$$

If true, then

$$\pi_1 R^{j-2} A_0 + \pi_1 R^{j-1} A_1 + \pi_1 R^j A_2 = 0, \quad j = 2, \ldots$$

or

$$A_0 + RA_1 + R^2 A_2 = 0$$
$M/H_2/1$ Queue

Two roots: matrix of all ones and a second matrix. If system is ergodic, this second matrix is correct solution and has spectral radius less than one (similar to $\rho < 1$ in $M/M/1$ queue) i.e., all eigenvalues are less than one.

We will call resulting $R$ the rate matrix. To get $\pi_0$ and $\pi_1$, need to solve

\[
\begin{align*}
\pi_0 B_{0,0} + \pi_1 B_{1,0} &= 0 \\
\pi_0 B_{0,1} + \pi_1 A_1 + \pi_2 A_2 &= 0
\end{align*}
\]

or

\[
(\pi_0, \pi_1) \begin{bmatrix} B_{0,0} & B_{0,1} \\ B_{1,0} & A_1 + RA_2 \end{bmatrix} = [0 \ 0]
\]

Also need normalization condition,

\[
1 = \pi_0 e + \pi_1 \sum_{j=1}^{\infty} R^{j-1} e = \pi_0 e + \pi_1 (I - R)^{-1} e
\]

where $e$ is a column vector (of appropriate size) with all ones and $I$ is the identity matrix.
$M/H_2/1$ Queue

**Q:** how to incorporate normalization condition into matrix equations?

Replace leftmost column with normalization condition on lhs and leftmost 0 with 1 on rhs,

$$
\begin{pmatrix}
\pi_0, \pi_1 \\
(I - R)^{-1}e & A_1 + RA_2
\end{pmatrix}
\begin{pmatrix}
1 \\
B_{0,1}
\end{pmatrix}
= [1 \ 0]
$$

**Q:** what is $E[N_q]$?

$$
E[N_q] = \sum_{j=1}^{\infty} (j - 1)\pi_j e = \pi_1 \sum_{j=1}^{\infty} (j - 1)R^{j-1}e
$$

$$
= \pi_1 R(I - R)^{-2}e
$$
**M/H\(_2\)/1 Queue**

Q: how to solve for \( R \)?

- use iterative procedure,

\[
R(0) = 0
\]
\[
R(n + 1) = -A_0 A_1^{-1} - R(n) A_2 A_1^{-1}
\]

- if system ergodic, guaranteed to converge
- there are other more efficient techniques

Q: when is system stable (ergodic)?

Calculate *expected drift* of repeating portion

- assume \( A_i \) are \( m \times m \) matrices

- define \( A = A_0 + A_1 + A_2 \) - can be interpreted as an infinitesimal generator for an MC that describes behavior of states within a level far, far to the right. Let \( f = (f_1, \ldots, f_m) \) be solution of

\[
fA = 0
\]

- stability condition is

\[
\text{drift to right} \quad < \quad \text{drift to left}
\]
\[
fA_0 e \quad < \quad fA_2 e
\]
General Case

- this is example of a quasi birth death (QBD) process

- general matrix geometric solution

\[ Q = \begin{bmatrix}
B_{0,0} & B_{0,1} & 0 & 0 & 0 & \cdots \\
B_{1,0} & B_{1,1} & A_0 & 0 & 0 & \cdots \\
B_{2,0} & B_{2,1} & A_1 & A_0 & 0 & \cdots \\
B_{3,0} & B_{3,1} & A_2 & A_1 & A_0 & \cdots \\
B_{4,0} & B_{4,1} & A_3 & A_2 & A_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{bmatrix} \]

\( B_{0,0} \) is \( m' \times m' \), \( B_{0,1} \) is \( m' \times m \), \( B_{i,0} \) is \( m \times m' \), and all others are \( M \times m \) matrices

- states divide into levels, \( m' \) in level 0, \( m \) in levels \( i = 1, 2, \ldots \) call states in level \( i \), \((i, j)\)

- \( \pi = (\pi_0, \pi_1, \ldots) \), where \( \pi_0 = (\pi_{0,1}, \ldots \pi_{0,m'}) \) and \( \pi_j = (\pi_{i,1}, \ldots, \pi_{i,m}) \), \( j = 1, \ldots \) and

\[ \pi Q = 0 \]
General Case

• balance equation for repeating portion

\[ \sum_{k=0}^{\infty} \pi_{j-1+k} A_k = 0, \quad j = 2, 3, \ldots \]

with solution

\[ \pi_j = \pi_1 R^{j-1}, \quad j = 2, 3, \ldots \]

• \( R \) is solution of

\[ \sum_{k=0}^{\infty} R^k A_k = 0 \]

• solution to boundary states

\[ (\pi_0, \pi_1) \begin{bmatrix} B_{0,0} & B_{0,1} \\ \sum_{k=1}^{\infty} R^{k-1} B_{k,0} & \sum_{k=1}^{\infty} R^{k-1} B_{k,1} \end{bmatrix} = 0 \]
General Case

- Normalization condition yields

\[
\begin{pmatrix} \pi_0, \pi_1 \end{pmatrix} \times \begin{bmatrix}
\frac{1}{(I - R)^{-1}e} & B_{0,0}^* \sum_{k=1}^{\infty} R^{k-1} B_{k,0} \\
B_{0,1} \sum_{k=1}^{\infty} R^{k-1} B_{k,1}
\end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

where \( A^* \) is matrix \( A \) with leftmost column removed.

- Calculation of \( R \)

\[
R(0) = 0
\]

\[
R(n + 1) = - \sum_{k \neq 1} R^k(n) A_k A_1^{-1}
\]

- When is system stable? Look at expected drift of process in repeating portion
  - Define \( A = \sum_{k=0}^{\infty} \) and \( f \) such that \( fA = 0 \)
  - Compute drift to right and to left; condition of stability is

\[
\text{drift to right} \quad < \quad \text{drift to left}
\]

\[
f A_0 e \quad < \quad f \sum_{k=2}^{\infty} (k - 1) A_k e
\]
Application to Multiprocessor with Failures

- 2 servers serving infinite capacity queue
- Poisson arrivals, \( \lambda \), exponential service times, \( \mu \)
- time between failures for single processor exponentially distr. with mean \( 1/\alpha \)
- single repairman, repair time exponentially distr. with mean \( 1/\gamma \)
- state of MC - \((n, u)\), \(n\) - no. of jobs in system; \(u\) - no. of processors operational
- let \( \pi = (\pi_0, \pi_1, \ldots) \); \( \pi_i = (\pi_{i,0}, \pi_{i,1}, \pi_{i,2}) \), \(i = 1, 2, \ldots\)
- infinitesimal generator

\[
Q = \begin{bmatrix}
B_{0,0} & A_0 & 0 & \cdots \\
B_{1,0} & B_{1,1} & A_0 & 0 & \cdots \\
0 & A_2 & A_1 & A_0 & \cdots \\
0 & 0 & A_2 & A_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
Application to Multiprocessor with Failures

\[
B_{0,0} = \begin{bmatrix}
-(\lambda + \gamma) & \gamma & 0 \\
\alpha & -(\lambda + \gamma + \alpha) & \gamma \\
0 & 2\alpha & -(\lambda + 2\alpha)
\end{bmatrix}
\]

\[
B_{1,1} = \begin{bmatrix}
-(\lambda + \gamma) & \gamma & 0 \\
\alpha & -(\lambda + \gamma + \alpha + \mu) & \gamma \\
0 & 2\alpha & -(\lambda + 2\alpha + \mu)
\end{bmatrix}
\]

\[
A_1 = \begin{bmatrix}
-(\lambda + \gamma) & \gamma & 0 \\
\alpha & -(\lambda + \gamma + \alpha + \mu) & \gamma \\
0 & 2\alpha & -(\lambda + 2\alpha + 2\mu)
\end{bmatrix}
\]

\[A_0 = \lambda I \quad A_2 = \text{diag}(0, \mu, 2\mu), \quad B_{1,0} = \text{diag}(0, \mu, \mu)\]

Can solve for $\pi$ using matrix geometric technique
Application to Multiprocessor with Failures

Let $I$ denote the number of processors that are operational, $I = 0, 1, 2$.

**Q:** what is $P(I = 0)$

$$P(I = 0) = \sum_{j=0}^{\infty} \pi_j \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

**Q:** what are conditions for stability?

$$A = A_0 + A_1 + A_2 = \begin{bmatrix} -\gamma & \gamma & 0 \\ \alpha & - (\gamma + \alpha) & \gamma \\ 0 & 2\alpha & -2\alpha \end{bmatrix}$$

Let $f = (f_1, f_2, f_3)$ be solution to

$$fA = 0 \quad \text{and} \quad fe = 1$$

where

$$e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
Stability Condition

Stability condition is:

\[ fA_0 e < fA_2 e \]

or

\[ \lambda < \frac{\mu \gamma(\gamma + 2\alpha)}{2\alpha^2 + 2\gamma \alpha + \gamma^2} \]
Phase Type Distributions

Let $S$ be an rv with a phase-type distr. Associated with this distr. is a $K + 1$ state continuous time MC where $K$ ($k = 1, \ldots, K$) are transient and one ($K + 1$) is an absorbing state with initial distribution $\pi_k(0) = P(X(0) = k)$, $k = 1, \ldots, K + 1$. By absorbing, I mean

- $\lambda_{K+1,j} = 0, j \neq K + 1$
- $\pi_{K+1}(t) \to 1$ as $t \to \infty$

$S$ is defined to be the time needed to reach $K$.

Examples:

- Erlang of order $r$: $K = r$, $\lambda_{i,i+1} = r\mu$, $i = 1, \ldots, r$, all other rates are zero. $\pi_1(0) = 1$, $\pi_i(0) = 0$, $i \neq 1$

- $H_2$ distr.: $K = 2$, $\lambda_{i,3} = \mu_i$, $i = 1, 2$, all other rates are zero. $\pi_i(0) = \alpha_i$, $i = 1, 2$, $\pi_3(0) = 0$
$M/PH/1$ Queue

- Poisson arrivals - $\lambda$

- Phase-type distr. - $K + 1$ states, transition rates $\lambda_{ij}$, and initial distr. $\alpha_i$

- system state - $(n, s)$ $n$ - no. of jobs in system, $s$ state of MC associated with PH distr.

- $Q$ can be expressed as

$$Q = \begin{bmatrix}
B_{0,0} & B_{0,1} & 0 & \cdots \\
B_{1,0} & B_{1,1} & A_0 & 0 & \cdots \\
0 & A_2 & A_1 & A_0 & \cdots \\
0 & 0 & A_2 & A_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$
\[M/PH/1 \text{ Queue}\]

\[B_{0,0} = [\lambda^2 - \lambda(1 - \alpha_{K+1})]\]

\[B_{0,1} = \lambda [\alpha_1 \ldots \alpha_K]\]

\[B_{1,0} = [\lambda_{1,K+1} \ldots \lambda_{K,K+1}]^T\]

\[A_0 = \text{diag}(\lambda, \ldots, \lambda)\]

\[A_1 = \begin{bmatrix}
-a_1 & \lambda_{1,2} & \cdots & \lambda_{1,K} \\
\lambda_{2,1} & -a_2 & \cdots & \lambda_{2,K} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{K,1} & \lambda_{K,2} & \cdots & -a_K
\end{bmatrix}\]

where \(a_i = \lambda + \sum_{k \neq i} \lambda_{i,k}\).

\[A_2 = \begin{bmatrix}
\lambda_{1,K+1}\alpha_1 & \lambda_{1,K+1}\alpha_2 & \cdots & \lambda_{1,K+1}\alpha_K \\
\lambda_{2,K+1}\alpha_1 & \lambda_{2,K+1}\alpha_2 & \cdots & \lambda_{2,K+1}\alpha_K \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{K,K+1}\alpha_1 & \lambda_{K,K+1}\alpha_2 & \cdots & \lambda_{K,K+1}\alpha_K
\end{bmatrix}\]

Solution proceeds as before.
Markov Modulated Poisson Process (MMPP)

Given a $K$ state continuous time MC $\{X(t)\}$ with transition rates $\{\lambda_{i,j}\}$. Given $K$ different arrival rates $\{\lambda_k\}$.

Arrival process is Poisson with rate $\lambda_{X(t)}$, $t \geq 0$, i.e., the MC $\{X(t)\}$ *modulates* the arrival rate.

**Important** uses in modeling bursty traffic sources in networks.
\( \text{MMPP}/M/1 \text{ Queue} \)

- arrivals according to a MMPP, transition rates \( \{\lambda_{i,j}\} \) and arrival rates \( \{\lambda_k\} \)
- exponential service times, \( \mu \)
- system state - \((n, s)\) \( n \) - no. of jobs in system, \( s \) state of MC modulating arrivals
- \( Q \) can be expressed as

\[
Q = \begin{bmatrix}
B_{0,0} & A_0 & 0 & \cdots \\
A_2 & A_1 & A_0 & 0 & \cdots \\
0 & A_2 & A_1 & A_0 & \cdots \\
0 & 0 & A_2 & A_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\]
**$MMPP/M/1$ Queue**

\[
B_{0,0} = \begin{bmatrix}
-a_1 & \lambda_{1,2} & \cdots & \lambda_{1,K} \\
\lambda_{2,1} & -a_2 & \cdots & \lambda_{2,K} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{K,1} & \lambda_{K,2} & \cdots & -a_K
\end{bmatrix}
\]

where $a_i = \lambda_i + \sum_{k \neq i} \lambda_{i,k}$

\[
A_1 = \begin{bmatrix}
-(a_1 + \mu) & \lambda_{1,2} & \cdots & \lambda_{1,K} \\
\lambda_{2,1} & -(a_2 + \mu) & \cdots & \lambda_{2,K} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{K,1} & \lambda_{K,2} & \cdots & -(a_K + \mu)
\end{bmatrix}
\]

\[
A_2 = \text{diag}(\mu, \ldots, \mu)
\]

\[
A_0 = \text{diag}(\lambda_1, \ldots, \lambda_K)
\]

Solution proceeds as before.