M/G/1 Processor Sharing Queue

- Poisson arrival process, rate \( \lambda \)
- i.i.d. service times \( \{X_i\} \), with density function \( f_X(x) \), cumulative distribution \( F_X(x) \), mean \( E[X] \).
- \( \{N(t)\} \), queue length process

Let state of system be \( \mathbf{X}(t) = (X_1(t), \ldots, X_{N(t)}) \) where \( X_i(t) \) is remaining service time of \( i \)-th customer in system.

Let \( f_n(t, x_1, \ldots, x_n) \) denote the joint probability of \( N(t) = n \), and density function for remaining times of customers being \( \mathbf{X}(t) = (x_1, \ldots, x_n) \).
Behavior of $f_n(t, x_1, \ldots, x_n)$

Approach: write expression for $f_n(t + \Delta t, x_1, \ldots, x_n)$ in terms of $f_m(t, \cdot)$.

\[
f_n(t + \Delta t, x_1, \ldots, x_n) = (1 - \lambda \Delta t)f_n(t, x_1 + \Delta t/n, \ldots, x_n + \Delta t/n) \]
\[
+ \sum_{i=0}^{n} \int_{0}^{\Delta t/n+1} f_{n+1}(t, x_1 + \frac{\Delta t}{n+1}, \ldots, x_{i-1} + \frac{\Delta t}{n+1}, y, x_i + \frac{\Delta t}{n+1}, \ldots, x_n + \frac{\Delta t}{n+1})dy \]
\[
+ \lambda \Delta t \sum_{i=1}^{n} f_X(x_i)f_{n-1}(x_1 + \frac{\Delta t}{n-1}, \ldots, x_{i-1} + \frac{\Delta t}{n-1}, x_{i+1} + \frac{\Delta t}{n-1}, \ldots, x_n + \frac{\Delta t}{n-1})
\]
Some equalities

\[ f_n(t, x_1 + \Delta t/n, \ldots, x_n + \Delta t/n) = f_n(t, x_1, \ldots, x_n) \]
\[ + \sum_{i=1}^{n} \frac{1}{n} \frac{\partial f_n(t, x_1, \ldots, x_n)}{\partial x_i} \Delta t + o(\Delta t) \]

\[ \int_{0}^{\frac{\Delta t}{n+1}} f_{n+1}(t, x_1 + \frac{\Delta t}{n+1}, \ldots, x_{i-1} + \frac{\Delta t}{n+1}, y, \]
\[ x_i + \frac{\Delta t}{n+1}, \ldots, x_n + \frac{\Delta t}{n+1})dy \]
\[ = f_{n+1}(t, x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n) \frac{\Delta t}{n+1} \]
\[ + o(\Delta t) \]

\[ \Delta t f_{n-1}(x_1 + \frac{\Delta t}{n-1}, \ldots, x_{i-1} + \frac{\Delta t}{n-1}, \]
\[ x_{i+1} + \frac{\Delta t}{n-1}, \ldots, x_n + \frac{\Delta t}{n-1}) \]
\[ = \Delta t f_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \]
\[ + o(\Delta t) \]
Behavior of $f_n(t, x_1, \ldots, x_n)$

Substituting the above into the expression for $f_n(t + \Delta, \cdots)$ yields,

$$f_n(t + \Delta t, x_1, \ldots, x_n) =$$

$$(1 - \lambda \Delta t)f_n(t, x_1, \ldots, x_n)$$

$$+ \lambda \Delta t \sum_{i=1}^{n} \frac{1}{n} \frac{\partial f_n}{\partial x_i}$$

$$+ \sum_{i=0}^{n} f_{n+1}(t, x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n) \frac{\Delta t}{n + 1}$$

$$+ \lambda \Delta t \sum_{i=1}^{n} f_X(x_i)f_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

$$+ o(\Delta t)$$
Behavior of $f_n(t, x_1, \ldots, x_n)$

Subtracting $f_n(t, x_1, \ldots, x_n)$ from both sides, dividing by $\Delta t$ and letting $\Delta t \to \infty$ yields

$$\frac{\partial f_n}{\partial t} =$$

$$\sum_{i=0}^{n} \frac{1}{n + 1} f_{n+1}(t, x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)$$

$$- \lambda f_n(t, x_1, \ldots, x_n)$$

$$+ \lambda \sum_{i=1}^{n} f_X(x_i) f_{n-1}(t, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

$$+ \lambda \sum_{i=1}^{n} \frac{1}{n} \frac{\partial f_n}{\partial x_i}$$
Steady state

Assume as $t \to \infty$, $f_n(t, x_1, \ldots, x_n) \to f_n(x_1, \ldots, x_n)$. Previous equations reduce to

$$0 = \sum_{i=0}^{n} \frac{1}{n+1} f_{n+1}(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)$$

$$- \lambda f_n(x_1, \ldots, x_n)$$

$$+ \lambda \sum_{i=1}^{n} f_X(x_i) f_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

$$+ \lambda \sum_{i=1}^{n} \frac{1}{n} \frac{\partial f_n}{\partial x_i}$$
Steady state solution

Steady state solution is

\[ f_n(x_1, \ldots, f x_n) = (1 - \rho)\lambda^n \prod_{i=1}^{n} (1 - F_X(x_i)) \]

Note that

\[ f_{n+1}(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n) \]

\[ = (1 - \rho)\lambda^{n+1} \prod_{i=1}^{n} (1 - F_X(x_i))(1 - F_X(0)) \]

\[ = \lambda f_n(x_1, \ldots, x_n) \]

and

\[ \frac{\partial f_n}{\partial x_i} = -(1 - \rho)\lambda^n f_X(x_i) \prod_{j \neq i} (1 - F_X(x_j)) \]

\[ = \lambda f_X(x_i) f_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \]
Steady state solution

Previous equalities can be used to establish steady state solution for $f_n$. Let $N(t) \to N$ as $t \to \infty$.

Q: what is $P(N = n)$?

$$P(N = n) = (1 - \rho)\lambda^n \prod_{i=1}^{n} \int_0^\infty (1 - F_X(x_i))dx_i$$

$$= (1 - \rho)\lambda^n \prod_{i=1}^{n} E[X]$$

$$= (1 - \rho)\rho^n$$
Notes

1. same queue length distribution as FCFS M/M/1 queue

2. insensitivity to distribution except to $E[X]$ 

Q: what is $E[T|X = x]$ 

$$E[T|X = x] = x(E[N] + 1)$$

$$x/(1 - \rho)$$