M/G/1 Processor Sharing Queue

- Poisson arrival process, rate $\lambda$
- i.i.d. service times $\{X_i\}$, with density function $f_X(x)$, cumulative distribution $F_X(x)$, mean $E[X]$.
- $\{N(t)\}$, queue length process

Let state of system be $X(t) = \{X_1(t), \ldots, X_{N(t)}\}$ where $X_i(t)$ is remaining service time of $i$-th customer in system.
M/G/1 Processor Sharing Queue

Let $f_n(t, x_1, \ldots, x_n)$ denote the joint probability of $N(t) = n$, and density function for remaining times of customers being $X(t) = (x_1, \ldots, x_n)$,

$$P(N(t) = n, X_1(t) < x_1, \ldots, X_n(t) < x_n) = \int_0^{x_1} \cdots \int_0^{x_n} f_n(t, y_1, \ldots, y_n) dy_n \cdots dy_1$$

Approach: write expression for $f_n(t + \Delta t, x_1, \ldots, x_n)$ in terms of $f_m(t, \cdot)$. 
\[ f_n(t + \Delta t, x_1, \ldots, x_n) \]
\[ = (1 - \lambda \Delta t) f_n(t, x_1 + \Delta t/n, \ldots, x_n + \Delta t/n) \]
\[ + \sum_{i=0}^{n} \int_0^{\frac{\Delta t}{n+1}} f_{n+1}(t, x_1 + \frac{\Delta t}{n+1}, \ldots, x_{i-1} + \frac{\Delta t}{n+1}, y, \]
\[ x_i + \frac{\Delta t}{n+1}, \ldots, x_n + \frac{\Delta t}{n+1} ) dy \]
\[ + \lambda \Delta t \sum_{i=1}^{n} f_X(x_i) f_{n-1}(x_1 + \frac{\Delta t}{n-1}, \ldots, x_{i-1} + \frac{\Delta t}{n-1}, \]
\[ x_{i+1} + \frac{\Delta t}{n-1}, \ldots, x_n + \frac{\Delta t}{n-1} ) \]
Some equalities
\[ f_n(t, x_1 + \Delta t/n, \ldots, x_n + \Delta t/n) = f_n(t, x_1, \ldots, x_n) \]
\[
+ \sum_{i=1}^{n} \frac{1}{n} \frac{\partial f_n(t, x_1, \ldots, x_n)}{\partial x_i} \Delta t + o(\Delta t)
\]

\[
\int_0^{\frac{\Delta t}{n+1}} f_{n+1}(t, x_1 + \frac{\Delta t}{n+1}, \ldots, x_{i-1} + \frac{\Delta t}{n+1}, y, x_i + \frac{\Delta t}{n+1}, \ldots, x_n + \frac{\Delta t}{n+1}) dy
\]

\[= f_{n+1}(t, x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n) \frac{\Delta t}{n+1} + o(\Delta t) \]
\[ \Delta tf_{n-1}(x_1 + \frac{\Delta t}{n-1}, \ldots, x_{i-1} + \frac{\Delta t}{n-1}, x_{i+1} + \frac{\Delta t}{n-1}, \ldots, x_n + \frac{\Delta t}{n-1}) \]

\[ = \Delta tf_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) + o(\Delta t) \]
Behavior of $f_n(t, x_1, \ldots, x_n)$

Substituting the above into the expression for $f_n(t + \Delta, \cdots)$ yields,

$$f_n(t + \Delta t, x_1, \ldots, x_n) =$$

$$(1 - \lambda \Delta t)f_n(t, x_1, \ldots, x_n) + \lambda \Delta t \sum_{i=1}^{n} \frac{1}{n} \frac{\partial f_n}{\partial x_i}$$

$$+ \sum_{i=0}^{n} f_{n+1}(t, x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n) \frac{\Delta t}{n + 1}$$

$$+ \lambda \Delta t \sum_{i=1}^{n} f_X(x_i) f_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) + o(\Delta t)$$
Behavior of $f_n(t, x_1, \ldots, x_n)$

Subtracting $f_n(t, x_1, \ldots, x_n)$ from both sides, dividing by $\Delta t$ and letting $\Delta t \to \infty$ yields

$$\frac{\partial f_n}{\partial t} = \sum_{i=0}^{n} \frac{1}{n + 1} f_{n+1}(t, x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)$$

$$- \lambda f_n(t, x_1, \ldots, x_n)$$

$$+ \lambda \sum_{i=1}^{n} f_X(x_i) f_{n-1}(t, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

$$+ \lambda \sum_{i=1}^{n} \frac{1}{n} \frac{\partial f_n}{\partial x_i}$$
Steady state

Now Assume as \( t \to \infty \), \( f_n(t, x_1, \ldots, x_n) \to f_n(x_1, \ldots, x_n) \).

Previous equations reduce to

\[
0 = \sum_{i=0}^{n} \frac{1}{n+1} f_{n+1}(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n) - \lambda f_n(x_1, \ldots, x_n) \\
+ \lambda \sum_{i=1}^{n} f_X(x_i) f_{n-1}(x_1, \ldots, x_{i-1}, x_i+1, \ldots, x_n) \\
+ \lambda \sum_{i=1}^{n} \frac{1}{n} \frac{\partial f_n}{\partial x_i}
\]
Steady state solution

Steady state solution is

\[ f_n(x_1, \ldots, x_n) = (1 - \rho) \lambda^n \prod_{i=1}^{n} (1 - F_X(x_i)) \]

Note that

\[ f_{n+1}(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n) \]

\[ = (1 - \rho) \lambda^{n+1} \prod_{i=1}^{n} (1 - F_X(x_i))(1 - F_X(0)) \]

\[ = \lambda f_n(x_1, \ldots, x_n) \]
and

\[
\frac{\partial f_n}{\partial x_i} = -(1 - \rho) \lambda^n f_X(x_i) \prod_{j \neq i} (1 - F_X(x_j))
\]

\[
= \lambda f_X(x_i) f_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)
\]

These equalities can be used to establish the correctness of the solution.
Steady state solution

Now Let \( N(t) \to N \) as \( t \to \infty \).

Q: what is \( P(N = n) \)?

\[
P(N = n) = (1 - \rho)\lambda^n \prod_{i=1}^{n} \int_{0}^{\infty} (1 - F_X(x_i)) \, dx_i
\]

\[
= (1 - \rho)\lambda^n \prod_{i=1}^{n} E[X]
\]

\[
= (1 - \rho)\rho^n
\]
Notes

1. same queue length distribution as FCFS M/M/1 queue

2. insensitivity to distribution except to $E[X]$

Q: what is $E[T|X = x]$?

$$E[T|X = x] = x(E[N] + 1)$$

$$= x/(1 - \rho)$$