Stochastic Ordering

Stochastic Ordering is useful for allowing an analyst to qualitatively compare systems

- under different workloads
- under different scheduling, routing algorithms
- \( X, Y \in \mathbb{R} \)
- \( F \) set of functions \( f : \mathbb{R} \to \mathbb{R} \)
  \[ F_{inc} = \{ f : \mathbb{R} \to \mathbb{R} | f \text{ increasing} \} \]
  \[ F_{cx} = \{ f : \mathbb{R} \to \mathbb{R} | f \text{ convex} \} \]
  \[ F_{icx} = \{ f : \mathbb{R} \to \mathbb{R} | f \text{ increasing, convex} \} \]

**Defn.** \( X \) is larger than \( Y \) in the sense of \( F \) (written \( Y \leq_F X \) iff

\[
E[f(Y)] \leq E[f(X)], \quad \forall f \in F
\]
$\mathcal{F}_{inc}$/strong ordering

Consider $\mathcal{F}_{inc}$ ordering.

**Thm.** Take $X, Y \in \mathbb{R}$, $Y \leq_{\mathcal{F}_{inc}} X$ iff $F_X(x) \leq F_Y(x)$, $\forall x \in \mathbb{R}$.

We’ll show $Y \leq_{\mathcal{F}_{inc}} X$ iff $F_Y(x) \Rightarrow F_X(x)$, $\forall x \in \mathbb{R}$.

Take $f(x) = -\mathbbm{1}(x < y)$, $y \in \mathbb{R}$. $f \in \mathcal{F}$. Therefore,

$$P(Y < y) = E[\mathbbm{1}(X < y)] \geq E[\mathbbm{1}(Y < y)] = P(X < y)$$

The student is encouraged to establish the $\Leftarrow$ part.
Vectorial stochastic orderings

Let $X, Y \in \mathbb{R}^n$, $n$ a nonnegative integer, $F$ such that $f : \mathbb{R}^n \to \mathbb{R}$. Then, previous definition still holds.

Henceforth, we focus on $\leq_{\mathcal{F}_{inc}}$ which we will abbreviate as $\leq_{st}$.

**Comment:** If $X, Y \in S \subset \mathbb{R}$, then we only care about the behavior of $f \in \mathcal{F}$ on $S$. Example: if $X, Y \geq 0$, then $f(x) = x^k$ is increasing for $k = 1, 2, \ldots$ whereas if $X, Y \in \mathbb{R}$, only $k = 1, 3, 5, \ldots$ is allowed.
Properties of $\leq_{st}$

**Property 1.** If $X_1, X_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \mathbb{R}$ are pairs of independent rv’s, and $Y_i \leq_{st} X_i$, $i = 1, 2$, then

$(Y_1, Y_2) \leq_{st} (X_1, X_2)$.

**Property 2.** If $X, Y \in \mathbb{R}^n$, and $Y \leq_{st} X$, then

$(f_1(Y), f_2(Y), \ldots, f_m(Y)) \leq_{st}$

$(f_1(X), f_2(X), \ldots, f_m(X))$, for any $f_1, f_2, \ldots, f_m \in \mathcal{F}_{inc}$.
Statistical multiplexer revisited

Recall that a statistical multiplexer is a discrete time system. It can serve one packet per time unit. \( \{A_n\} \) is a sequence of iid rv’s. Last, let \( \{X_n\} \) denote the queue length process (number of packets in multiplexer at each time unit). \( X_n \) satisfies the following relation,

\[
X_{n+1} = (X_n - 1)^+ + A_n, \quad n = 0, 1, \ldots.
\]

Note that the rhs is an increasing function in \( X_n \) and \( A_n \).

Consider two statistical multiplexers that differ in their arrival processes, \( \{A^1_n\} \) and \( \{A^2_n\} \) where \( A^1_n \leq_{st} A^2_n \).
Workload comparison

**Thm.** If \( \{X^1_n\} \) and \( \{X^2_n\} \) denote the queue lengths for these two systems, then \( X^1_n \leq_{st} X^2_n, \ n = 1, \ldots \) provided that \( X^1_0 \leq_{st} X^2_0. \)

**Proof.** By induction on \( n. \)

**Basis step.** By assumption, \( X^1_0 \leq_{st} X^2_0. \)

**Inductive hypothesis.** Assume that \( X^1_j \leq_{st} X^2_j, \ j = 0, \ldots, n. \) Because \( \{A^i_k\} \) are iid sequences, we can apply Property 1 to yield

\[
(X^1_n, A^1_n) \leq_{st} (X^2_n, A^2, n)
\]

Now,

\[
X^1_{n+1} = (X^1_n - 1)^+ + A^1_n
\leq_{st} (X^2_n - 1)^+ + A^2_n
= X^2_{n+1}
\]

where the inequality is due to Property 2.