Generating random variables

How do you generate instances of random variables?
• given $X$, $F_X(x)$, generate $x_1, x_2, x_3, \ldots$
• assume can generate numbers, uniformly distributed between 0 and 1 (ranf)
• inverse function method
  $x_1 = F^{-1}(\text{ranf})$
If $F_X(x) < F_Y(x)$, for all $x$, then

$$X <_d Y$$

$X$ is less than $Y$ in distribution.
Stochastic Processes
Stochastic process $X = \{X(t), t \in T\}$ is a collection of random variables (rvs)

- one rv for each $X(t)$ for each $t \in T$
- index set $T$ - possible set of values of $t$
- state space - possible set of values of $X(t)$
- if $T$ is countable, then $X$ is discrete space process, will use notation $X = \{X_n, n \in T\}$
- if $T$ is continuous, then $X = \{X(t), t \in T\}$ is continuous-time process
- $X(t)$ can take values from discrete- or continuous-state space
Examples

- no. transactions processed by a database system during interval \((0,t)\) - cont.-time, discrete-space
- no. packets thru router during \(n\)-th hour of day, \(\{X_n, n = 1,2,\ldots,24\}\), discrete-time, discrete-space
- response time of request to google server given that it arrives at time \(t\), \(\{X(t), t \geq 0\}\)
- Bernoulli process: \(\{Y_n, n = 0,1,\ldots\}\),

\[
P(Y_n = i) = \begin{cases} 
p, & i = 0, \\
1-p, & i = 1, \\
0, & i \neq 0,1. 
\end{cases}
\]
Counting process: a stochastic process that represents no. of events that occurred by time $t$; a continuous-time, discrete-state process $\{N(t), \ t \geq 0\}$

**Definition:** $\{N(t), \ t \geq 0\}$ is a counting process if

- $N(0) = 0$
- $N(t) \geq 0$
- $N(t)$ increasing (nondecreasing) in $t$
- $N(t) - N(s)$ is no. events in interval $[s, t]$
Counting process

- counting process has *independent increments* if no. events in disjoint intervals are independent

\[ P(N_1 = n_1, N_2 = n_2) = P(N_1 = n_1)P(N_2 = n_2) \]

- counting process has *stationary increments* if no. events in \([t_1 + s, t_2 + s]\) has the same distribution as no. events in \([t_1, t_2]\), \(s > 0\)
Bernoulli process

- \( N_i \) - no. of successes by time \( i = 0,1,... \) is a counting process with independent and stationary increments
  - \( p \) – probability of failure; \( 1-p \) – prob. of success

- \( P(N_i = n) = \binom{i}{n} (1-p)^n p^{i-n} \), \( n = 0,1,...,i \)

- \( E[N_i] = i (1 - p) \), \( \sigma^2_{N_i} = ip (1 - p) \), \( i = 0,1,... \)

- \( X \) - time between successes,
  - \( P(X = n) = (1 - p) p^{n-1} \), \( n = 1,2,... \)
  - \( F_X(n) = P(X < n) = 1 - p^n \), \( n = 1,2,... \)
  - \( E[X] = 1/(1 - p) \), \( \sigma^2_X = p/(1 - p)^2 \)
\[ P(N_i = n) = \binom{i}{n} (1-p)^n p^{i-n} \]
\[ P(X=n) = p^{n-1}(1 - p) \quad n = 1, \ldots \]
Bernoulli process

- $X^{(n)}$ - time between success and $n$-th successive success,

$$P(X^{(n)} = k) = \binom{k-1}{n-1} (1 - p)^n p^{k-n}, \quad k = n, n+1, \ldots$$

called Pascal distribution

$$E[X^{(n)}] = n/(1 - p)$$

- memoryless property

$$P(X = l + n | X > l) = (1 - p)p^{n-1}, \quad l \geq 0; n \geq 1$$
$X^{(n)}$
\[ P(X^{(n)} = k) = P(N_{n-1} = k-1) (1 - p) \quad k = 1, \ldots \]

\[ \binom{k-1}{n-1} (1 - p)^n p^{k-n}, \quad k = n, n+1, \ldots \]
**Little o Notation**

**Definition:** $f$ is $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

- $f(h) = h^2$ is $o(h)$
- $f(h) = h$ is not
- $f(h) = h^r, \ r > 1$ is $o(h)$
- $\sin(h)$ is not
- If $f, g$ are $o(h)$, then $f(h) + g(h) = o(h)$
Example

**Example:** exponential rv $X$ with parameter $\lambda$ has distribution $P(X < h) = 1 - e^{-\lambda h}$, $h > 0$,

\[
P(X \leq t + h \mid X > h) = P(X \leq h),
\]
\[
= 1 - e^{-\lambda h},
\]
\[
= 1 - [1 - \lambda h + \sum_{n=2}^{\infty} (-\lambda h)^n / n!]
\]
\[
= \lambda h + o(h)
\]
Poisson process

counting process \{N(t), \ t \geq 0\} with rate \( \lambda > 0 \)

- independent and stationary increments
- \( P(N(h) = 1) = \lambda \ h + o(h) \)
- \( P(N(h) \geq 2) = o(h) \)
  \[ \Rightarrow P(N(h) = 0) = (1- \lambda \ h) + o(h) \]

let \( P_n(t) = e^{-\lambda t} (\lambda t)^n / n! \), \( n = 0, 1, \ldots \)

- \( E[N(t)] = \lambda \ t \), \( \sigma^2_{N(t)} = \lambda \ t \)
- \( X \), time between events, \( F_X(t) = 1 - e^{-\lambda t} \), \( t \geq 0 \)
  probability density function (pdf) \( f_X(t) = \lambda e^{-\lambda t} \)
  \( E[X] = 1/\lambda \), \( \sigma^2_X = 1/\lambda^2 \)
\[ P_n(t+\Delta t) = P_{n-1}(t) \lambda \Delta t + P_n(t)(1- \lambda \Delta t) + o(\Delta t) \]

\[ P_n(t+\Delta t) - P_n(t) = P_{n-1}(t) \lambda \Delta t - P_n(t) \lambda \Delta t + o(\Delta t) \]

\[ \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = P_{n-1}(t) \lambda - P_n(t) \lambda \Delta t + o(\Delta t)/\Delta t \]

\[ \frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - \lambda P_n(t), \quad n=1,2,\ldots \]

\[ \frac{dP_0(t)}{dt} = -\lambda P_1(t) \]
Solutions

\[ \frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - \lambda P_n(t), \quad n = 1,2,\ldots \]
\[ \frac{dP_0(t)}{dt} = -\lambda P_1(t) \]

\[ P_n(t) = e^{-\lambda t} \left( \frac{\lambda t}{n!} \right), \quad n = 0,1,\ldots \]
Poisson process

- $X^{(n)}$, time from event until $n$-th successive event
  \[ f_{X^{(n)}}(t) = \lambda(\lambda t)^{n-1} e^{-\lambda t} / n!, \quad t \geq 0 \] (Erlang rv of order $n$)

- take iid sequence of exponential rvs with rate $\lambda$,
  \[ \{X_i\}_{i=1} \text{ define } N(t) = \max\{n| \sum_{1 \leq t \leq n} X_i \leq t\}, \]
  \[ \{N(t)\} \text{ is a Poisson process} \]
Poisson process

- if $N(t)$ is a Poisson process and one event occurs in $[0, t]$, then the time to the event is uniformly distributed in $[0, t]$,

\[ f_{X_1|N(t)=1}(x|1) = \frac{1}{t}, \quad 0 \leq x \leq t \]

- if $N_1(t)$ and $N_2(t)$ are independent Poisson processes with rates $\lambda_1$ and $\lambda_2$, then $N(t) = N_1(t) + N_2(t)$ is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$
Poisson process

- $N(t)$ is Poisson with rate $\lambda$, $M_i$ is Bernoulli with success prob. $p$. Construct a new process $L(t)$ by only counting the $n$-th event in $N(t)$ whenever $M_n > M_{n-1}$ (i.e., success at time $n$) $L(t)$ is Poisson with rate $\lambda p$

- exhibits memoryless property,

$$f_{X|X>t}(x|t) = \lambda e^{-\lambda(x-t)},$$

or if $X = t+Y$, i.e., $Y$ is the remaining time until event,

$$f_Y(y) = \lambda e^{-\lambda y} = f_X(y)$$
Example

Consider a web server where failures are described by a Poisson process with rate $\lambda = 2.4$/day, i.e., the time between failures, $X$, is exponential $rv$ with mean $E[X] = 10$ hrs.

- $P($time between failures $< T$ days$) = \quad$
- $P($k failures in $T$ days$) = \quad$
- $P(N(5) < 10) = \quad$
- Look in on system at random day, what is prob. of no. failures during next day?
- Failure is memory failure with prob. 1/9, CPU failure with prob. 8/9. Failures occur as independent events. What is process governing memory failures?
Review

- Bernoulli process \( \{N_i\} \) with parameter \( p \) (prob of event)
- Counting process with stationary and independent increments
  \[
P(N_i = n) = \binom{i}{n} p^n(1-p)^{i-n}
\]
- \( X \) – interevent time (interarrival time)
  \[
P(X = k) = (1-p)^{k-1} p
\]
- \( X^{(n)} \) – time to n-th successive event
Poisson process \{N(t)\} with parameter \(\lambda\) (event rate)

counting process with stationary and independent increments

\[ P(N(t) = n) = (\lambda t)^n e^{-\lambda t} / n! , \quad n=0,1,\ldots \]

\(X\) – interevent time (interarrival time)

\[ f_X(x) = \lambda e^{-\lambda t} , \quad t \geq 0 \]

\(X^{(n)}\) – time to \(n\)-th successive event
Properties

- Bernoulli and Poisson processes exhibit memoryless properties, i.e.,
  \[ F_{X|X>t}(x-t) = F_X(x-t) \]
- Sum of two Poisson processes with rates \( \lambda_1 \) and \( \lambda_2 \) is Poisson with rate \( \lambda_1 + \lambda_2 \)
- Thinning a Poisson process with rate \( \lambda \) using a Bernoulli process with probability \( p \) yields a Poisson process with rate \( \lambda p \)
Order Statistics

- $X_1, X_2, \ldots, X_n \in \mathbb{R}$
- $X_{(i)}$ – i-th smallest value of $X_1, X_2, \ldots, X_n$
  - i-th order statistic
- range $X_{ij} - X_{(i)} - X_{(j)}$
- examples
  - parallel execution of n independent tasks, $X_{(n)}$
  - fault tolerant computing, majority voting, $n = 3$, $X_{(2)}$
  - diversity routing, $X_{(1)}$
  - reliable multicast
Max, Min

- $F(x) \equiv F_{X_i}(x)$; $F_{i}(x) \equiv F_{X(i)}(x)$
- $F_n(x) = P(\max\{X_i\} \leq x)$
  - $= P(X_1 \leq x, \ldots, X_n \leq x)$
  - $= \prod P(X_i \leq x) = (F(x))^n$
- $F_1(x) = P(\min\{X_i\} \leq x)$
  - $= 1 - P(\min\{X_i\} > x)$
  - $= 1 - \prod P(X_i > x)$
  - $= 1 - (1-F(x))^n$
\( F_i(x) \)

\[ F_r(x) = P(X_{(r)} \leq x) \]

\[ = P(\text{at least } r \text{ of } X_i \text{ are less than } x) \]

\[ = \sum_{r \leq j \leq n} \binom{n}{j} (F(x))^j (1-F(x))^{n-j} \]
Consider a sender sending sequence of packets to $R$ receivers. Assume packet losses at receivers are described by identical and independent Bernoulli processes with loss prob. $p$. Suppose sender transmits each message repeatedly until correctly received at least once by each receiver. Let $M_i$ denote the number of transmissions needed to transmit $i$-th msg. correctly to all receivers. What are the statistics of $\{M_i\}$?
because of definition of Bernoulli process, $M_i = M$ independent of $i$

let $X_i$ denote the number of transmissions required for receiver $i$ to receive msg; $P(X_i \leq n) = 1-p^n$

note that $M = \max_{1 \leq i \leq R} X_i$

therefore, $P(M < n) = \prod P(X_i \leq n)$

$= (1 - p^n)^R$
What is $E[M]$?

whenever $X$ is a nonnegative rv,

$$E[X] = \int_0^\infty (1 - F_X(x)) \, dx$$

if $X$ is discrete

$$E[X] = \sum_{i=0}^{\infty} P(X > i)$$
therefore

\[ E[M] = \sum_{i=0}^{\infty} \left( 1 - \left(1 - p^i \right)^R \right), \]

\[ = \sum_{i=1}^{R} \binom{R}{i} (-1)^{i+1} \frac{1}{(1 - p^i)} \]
Reliable Multicast

Note slow growth. Suggests that reliable multicast protocol designs should be able to scale up in number of receivers.
Renewal process

- Counting process in which inter-event time $X_1, X_2, \ldots$ are iid rvs with cdf $F_X(x) = P(X \leq x)$, $x \geq 0$.

- Assume it also has density function $f_X(x)$, $x \geq 0$. 
Q. what is \( f_{X|X>y}(x|X>y) \) ?

A. 
\[
f_{X|X>y}(x \mid X > y) = \begin{cases} 
  f_X(x)/(1 - F_X(x)), & x > y, \\
  0, & \text{otherwise} 
\end{cases}
\]

if \( Y = X-y \), then

\[
f_{Y|X>y}(w \mid X > y) = \frac{f_X(w+y)}{1 - F_X(y)}
\]
Random incidence

- given renewal process with $f_X(x)$
- $Y$ – time from random point in time to next event.

$f_Y(x)$?
What is $f_W(w)$?

$$f_W(w) \propto w \ f_X(w)$$

$$= w \ f_X(w)/E[X], \ w \geq 0$$

What is $f_{Y|W}(y|w)$?

$$f_{Y|W}(y|w) = \begin{cases} 
1/w, & 0 \leq y \leq w \\
0, & \text{otherwise}
\end{cases}$$
\( f_{W,Y}(w,y) \)?

\[
f_{W,Y}(w,y) = f_{W}(w) \cdot f_{Y|W}(y|w)
\]

\[
= \begin{cases} 
  f_{X}(w)/E[X], & 0 \leq y \leq w \\
  0, & \text{otherwise}
\end{cases}
\]
\[ f_Y(y) = \int f_{W,Y}(w,y) \, dw \]

\[ = \int_{y}^{\infty} f_X(w) / E[X] \, dw \]

\[ = (1 - F_Y(y)) / E[X] \]
Remarks

- \( E[Y] = \frac{E[X]}{2} + \frac{\text{Var}[X]}{2E[X]} \)

- **Note:**
  - if \( \text{Var}[X] > E^2[X] \), then \( E[Y] > E[X] \)
  - if \( \text{Var}[X] = E^2[X] \), then \( E[Y] = E[X] \)
  - if \( \text{Var}[X] < E^2[X] \), then \( E[Y] < E[X] \)

- \( X - \text{exponential} \rightarrow f_Y(y) = \lambda e^{-\lambda y} \) (memoryless)
Birth Death process

Continuous-time, discrete-space stochastic process \( \{N(t), \ t \geq 0\} \), \( N(t) \in \{0, 1,...\} \)

\( N(t) \) - population at time \( t \)

- \( P(N(t+h) = n+1 \mid N(t) = n) = \lambda_n h + o(h) \)
- \( P(N(t+h) = n-1 \mid N(t) = n) = \mu_n h + o(h) \)
- \( P(N(t+h) = n \mid N(t) = n) = 1-(\lambda_n + \mu_n) h + o(h) \)
- \( \lambda_n \) - birth rates
- \( \mu_n \) - death rates, \( \mu_0 = 0 \)

Q: what is \( P_n(t) = P(N(t) = n) \)? \( n = 0,1,... \)
Birth Death process

\[ \frac{dP_n(t)}{dt} = P_{n-1}(t) \lambda_{n-1} + P_{n+1}(t) \mu_{n+1} - (\lambda_n + \mu_n) P_n(t), \quad n=1,\ldots \]

\[ \frac{dP_0(t)}{dt} = P_1(t) \mu_1 - \lambda_0 P_0(t) \]

- behavior of \( P_n(t) \) depends on initial condition \( P_n(0) \)
Stationary behavior of B-D process

Behavior as $t \rightarrow \infty$

if reasonable system, expect it to reach equilibrium

Q: what is equilibrium? no change in $P_n(t)$ as $t$ changes; does not depend on initial conditions

$$(\lambda_n + \mu_n) P_n = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}$$

where

$$P_n = \lim_{t \rightarrow \infty} P_n(t)$$
Transition state diagram:

Balance equations:

rate of transitions into $n = \text{rate of transitions out of } n$

$\lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1} = (\lambda_n + \mu_n) P_n, \quad n \geq 1$

$\mu_1 P_1 = \lambda_0 P_0, \quad n = 0$
B-D process

**Note:** have a countable no. linear equations; also need

\[ \sum_{n=0}^{\infty} P_n = 1 \]

**Example:** queueing system with one server, no waiting line
- Poisson arrivals, rate \( \lambda \)
- Exponential service times, rate \( \mu \)

**Solution:**

- \( P_0 = \frac{\mu}{\mu + \lambda} \)
- \( P_1 = \frac{\lambda}{\mu + \lambda} \)